The Quantum World

A Rapid Introduction: Day 1

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Introduction

This workshop is designed as a brief but mathematically rigorous introduction to quantum science and technology for the tech enthusiast. If at the end of this workshop you still feel bewildered by the perplexity that is quantum mechanics, rest in the comfort that even the best minds have struggled with attaining a true grasp of this aspect of reality.

"If quantum mechanics hasn't profoundly shocked you, you haven't understood it yet."

– Niels Bohr

Overview

1. Why Quantum Mechanics
2. A Dive into Superposition
3. Quantum Promotions

Why Quantum Mechanics: Applications

Figure: MRI (NMR)

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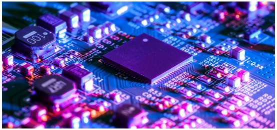

Figure: Processors

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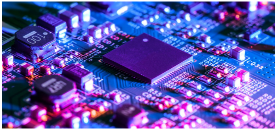

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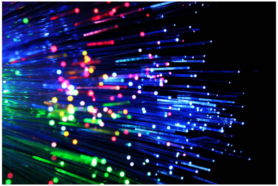

Figure: Fiber Optics

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- There are several experiments that pointed this out to the early pioneers of quantum theory.
- One of these experiments is the Young's double slit experiment which led to the notion of *wave-particle duality*.

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- As a result, the outgoing diffracted waves overlap, causing a superposition of wave amplitudes and bright and dark fringes on a detector.

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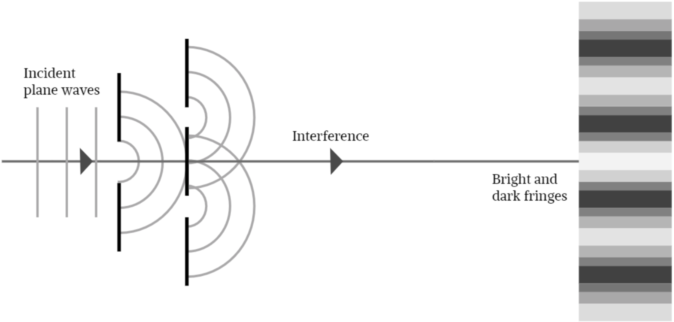

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- What was seen by Davisson and Germer was in fact an interference pattern on the screen, exactly as what Young did for light!
- This bizarre result gave rise to the notion of *wave-particle duality*, and matter could no longer be thought of deterministic chunks as they were believed to be in classical physics.

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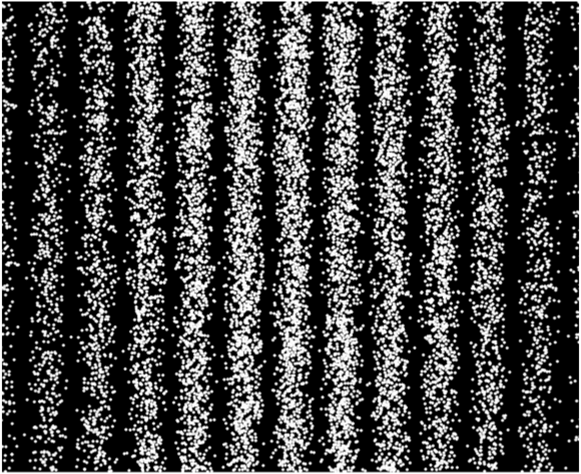

Davisson and Germer Double Slit Experiment

This result, along with the postulates of *photons* by **Einstein** and *matter waves* by **de Broglie**, breathed life into the following equations:

$$E = h\nu$$
$$p = \frac{h}{\lambda}$$

where E is the photon energy, ν is the frequency of light, λ is the wavelength of light and $h = 6.62 \times 10^{-34} Js$ is known as the *Planck's constant*, with its origins come from the *ultraviolet catastrophe*.

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- We will be looking at **Mach-Zehnder** Interferometry to discover and model superposition in a simple quantum system (2-level system).
- Interferometry is a class of experimental techniques that superimposes electromagnetic waves and exploits the properties of interference to gather information.

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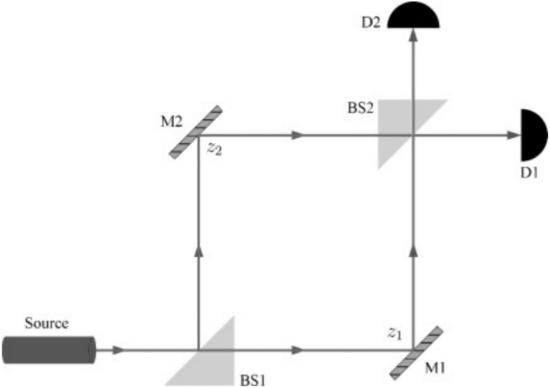

Mach-Zehnder Interferometer

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- To perform mathematical operations on these states, we can construct a *representation* of these states with vectors of a 2D complex vector space.

$$|u\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; |l\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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- This means that an arbitrary state in this vector space is written as:

$$|\psi\rangle = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \; \alpha, \beta \in \mathbb{C}$$

A Dive into Superposition: Matrix Representation

The action of the beam splitters and mirrors on the photons can be modelled as linear transformations (2×2 *matrix representations*) on $|\psi\rangle$.

Definition

Given a linear transformation $T : V \to U$ *and the bases for* V *and* U *being* $B_V = \{\vec{v}_j\}_{j=1}^{|V|}$ *and* $B_U = \{\vec{u}_j\}_{j=1}^{|U|}$ *respectively, then we have:*

$$T(\vec{v}_j) = \sum_i a_{ij}\vec{w}_i \tag{1}$$

where a_{ij} *are the entries of the* **matrix representation** *of* T *with respect to* B_V *and* B_U*.*

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- We have to enforce *unitarity* which allows $\left\| \left| \psi \right\rangle \right\|^2$ to be invariant as unity because of probability theory.

$$\left\| (BS) \left| \psi \right\rangle \right\|^2 = \left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|^2 = 1$$

$$\Rightarrow \ ((BS) \left| \psi \right\rangle)^\dagger (BS) \left| \psi \right\rangle = (\left| \psi \right\rangle)^\dagger (BS)^\dagger (BS) \left| \psi \right\rangle = 1$$

$$\Rightarrow \ \boxed{(BS)^\dagger (BS) = \mathbb{I}}$$

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- Any classical approach would fail with 100% certainty.
- An Elitzur-Vaidman bomb is a bomb with a photo-detector used as its trigger. If the bomb is working, a single photon incident on the photo-detector would cause the bomb to go off.

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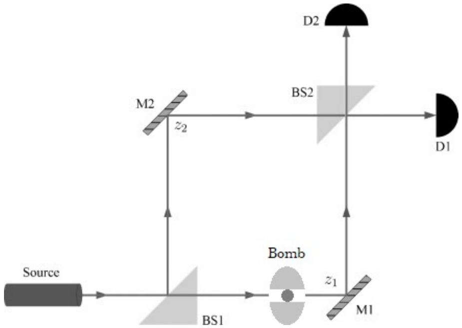

Figure: Elizur-Vaidman Bomb Detection Set-Up

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- Consider a photon entering the upper channel $|u\rangle$. After passing through the first beam splitter:

$$(BS1)\,|u\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

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- The 'split' photon continues to pass through BS2.

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- The result of a defective bomb is that we will **always** get a reading from D2.

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- Evidently if the photon does in fact enter the lower path, the bomb detonates and the experiment is undoubtedly over (not too great).
- But if the photon enters the upper path:

$$(BS2)\,|u\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \tag{2}$$

- Amazingly, we retrieve the same result with an equal probability for the beams to be detected by D1 and D2!

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Detector outcomes	$\mathbb{P}_{\text{defective}}$	$\mathbb{P}_{\text{working}}$
Photon enters D1	0	$1/4$
Photon enters D2	1	$1/4$
Bomb is detonated	0	$1/2$

Table: Elitzur-Vaidman Bomb Detection Outcomes

We are able to detect a working Elitzur-Vaidman obomb without detonating it with a $1/4$ probability.

Break

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- This nicely mirrors the unit-norm condition imposed on quantum states in the Mach-Zehnder interferometry experiment.

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- Taking a second derivative with respect to x:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) = \frac{p^2}{2m}\Psi(x,t) = E\Psi(x,t)$$

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Quantum Promotions: Activity 1

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(*Hint: Think about the simplest form of an operator that can act on a wave function to give you the position* x.)

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- Amazingly, taking both the time and 2 spatial derivatives extract the energy of the wave function!
- Putting these results together, we get the **free-particle Schrödinger equation** (FPSE):

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)$$

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- We extend the kinetic energy operator in the FPSE with the Hamiltonian to get the **time-dependent Schrödinger equation**:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t)$$

Thank you!

https://tinyurl.com/TQWday1