8.033 Relativity

Formula Sheet

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1 Hyperbolic Identities

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = i\sin(-ix)$$
$$\cosh(x) = cos(ix)$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$
$$\cosh^2(x) - \sinh^2(x) = 1$$
$$\cosh^2(x) = \frac{1}{\sqrt{1-\tanh^2(x)}}$$

2 Galilean Transformations

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v_x & 1 & 0 & 0 \\ -v_y & 0 & 1 & 0 \\ -v_z & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$
$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ +v_x & 1 & 0 & 0 \\ +v_y & 0 & 1 & 0 \\ +v_z & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}$$

3 Special Relativity

3.1 Invariants and Metric

$$\triangle s^2 = -\triangle t^2 + \triangle x^2 + \triangle y^2 + \triangle z^2$$
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

1. $\triangle s^2 < 0 \Rightarrow$ timelike separation
2. $\triangle s^2 > 0 \Rightarrow$ spacelike separation
3. $\triangle s^2 < 0 \Rightarrow$ null separation

Minkoski metric,

$$\eta^{\mu\nu} = \text{diag}(+1, 1, 1, 1)$$
$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$$

3.2 Lorentz Transformations

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 & 0 & 0 \\ -\gamma\beta_y & 0 & 1 & 0 \\ -\gamma\beta_z & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$
$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & +\gamma\beta_x & +\gamma\beta_y & +\gamma\beta_z \\ +\gamma\beta_x & 1 & 0 & 0 \\ +\gamma\beta_y & 0 & 1 & 0 \\ +\gamma\beta_z & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}$$
$$\text{where,} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}},\ \beta_i = \frac{v_i}{c}$$

As hyperbolic functions for unidirectional motion with *rapidity* η,

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh(\eta) & \sinh(\eta) \\ \sinh(\eta) & \cosh(\eta) \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$$

3.3 3-Vectors

3.3.1 3-Velocity

$$(\vec{u})^i = \frac{dx^i}{dt},\ (\vec{u})^i = \frac{u^i}{u^0}$$

a stationary observer sees a moving frame (S') with velocity v. There is another observer moving in S' with velocity u^x relative to S'. Both S' and u^x only have components in the x-direction.

$$u^x_{obs} = \frac{u^x + v}{1 + \frac{u^x v}{c^2}},\ u^{y,z}_{obs} = \frac{u^{y,z}}{\gamma_v(1 + \frac{u^{y,z}v}{c^2})}$$
$$\gamma_v = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

3.3.2 3-Momentum

$$\vec{p} = \gamma_v m\vec{v}$$
$$E^2 = |\vec{p}|^2 c^2 + m^2 c^4$$

3.3.3 3-Force

$$(\vec{F})^i = \frac{dp^i}{dt} = m\frac{d}{dt}(\gamma\vec{v}^i)$$

For a boost in the x-direction,

$$F^{x'} = \frac{F^x - v\vec{F}\cdot\vec{u}}{1 - u_x v},\ F^{y',z'} = \frac{F^{y,z}}{\gamma_v(1 - u_{y,z}v)}$$

where v is the primed frame velocity and $\vec{u}$ is the velocity of the object (relative to unprimed rest frame) experiencing the force.

3.4 4-Vectors

3.4.1 4-Velocity

$$u^\mu = \frac{dx^\mu(\tau)}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma_u\{c, u^x, u^y, u^z\}^\mu$$
$$u_\mu u^\mu = \eta_{\mu\nu}u^\nu u^\mu = -c^2$$

3.4.2 4-Momentum

$$p^\mu = mu^\mu = m\gamma_u\{c, u^x, u^y, u^z\}^\mu$$
$$p_\mu p^\mu = \eta_{\mu\nu}p^\nu p^\mu = -(mc)^2 = -\frac{E_{rest}}{m}$$
$$\gamma = \frac{E}{mc^2}, \text{ for photons, } p_\mu p^\mu = 0.$$

3.4.3 4-Acceleration and 4-Force

$$a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau}$$
$$a^\mu u_\mu = \frac{1}{2}\frac{\partial}{\partial\tau}(u_\mu u^\mu) = 0$$
$$\text{In the MCRF, } u^\mu_{MCRF} = \{1, 0, 0, 0\}^\mu$$
$$\Rightarrow a^\mu_{MCRF} = \{0, a_x, a_y, a_z\}^\mu$$
$$f^\mu = \frac{dp^\mu}{d\tau} = \gamma_u\{\vec{u}\cdot\vec{F}, \vec{F}\}^\mu$$

3.5 Uniform Accelerating Frames

For a uniform acceleration (g) w.r.t a stationary lab frame observer (assuming we start from $\tau = 0$),

$$x^\mu = \frac{1}{g}\{\sinh(g(\tau-\tau_0)), \cosh(g(\tau-\tau_0)) - 1, 0, 0\}^\mu$$
$$u^\mu = \{\cosh(g(\tau-\tau_0)), \sinh(g(\tau-\tau_0)), 0, 0\}^\mu$$
$$a^\mu = g\{\sinh(g(\tau-\tau_0)), \cosh(g(\tau-\tau_0)), 0, 0\}^\mu$$
$$\text{3-velocity} = v = \tanh(g(\tau-\tau_0))$$
$$\gamma = \frac{dt}{d\tau} = \cosh(g(\tau-\tau_0))$$
$$\text{In general, starting from } \tau = \tau_0,$$
$$\triangle x = \frac{1}{g}\cosh(g(\tau-\tau_0))\Big|_{\tau_1}^{\tau_2}$$
$$\triangle t = \frac{1}{g}\sinh(g(\tau-\tau_0))\Big|_{\tau_1}^{\tau_2}$$

3.6 Electromagnetism

3.6.1 Faraday Antisymmetric Tensor

$$F^{\mu\nu} = -F^{\nu\mu}$$
$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_X & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

3.6.2 Maxwell's Equations

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 j^\nu,\ j^\nu = \{\rho, \vec{J}\}^\nu}$$
$$\Rightarrow \vec{F} = q\vec{E} + q(\vec{v}\times\vec{B})$$
$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla\times\vec{B} = \mu_0\vec{J} + \frac{\partial}{\partial t}\vec{E}$$
$$\boxed{\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = 0^\sigma}$$
$$\Rightarrow \nabla\cdot\vec{B} = 0$$
$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

3.6.3 Natural Units

$$c = 1$$
$$\Rightarrow E = \gamma m$$

3.6.4 E/B field Transformations

$$F^{\alpha'\beta'} = \Lambda^{\alpha'}{}_{\mu}\Lambda^{\beta'}{}_{\nu}F^{\mu\nu} = \Lambda F \Lambda^T$$
$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}$$
$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B})$$
$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$$
$$\vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \vec{v} \times \vec{E})$$

$\parallel$ and $\perp$ are relative to boost,

3.6.5 Continuity Equations

$$\partial_\mu j^\mu = 0,\ j^\mu = \{\rho, \vec{J}\}^\mu$$

Number Density: $n^\mu = \gamma\{N_0, N_0\vec{u}\}^\mu$

$$N = \gamma N_0,\ \frac{\partial N}{\partial t} + \nabla \cdot \vec{N} = 0$$

3.7 Stress Energy tensors

$T^{\mu\nu} = T^{\nu\mu}$, where

1. $T^{00} \Rightarrow$ Energy Density
2. $T^{0i} \Rightarrow$ Energy Current Density
3. $T^{i0} \Rightarrow$ Momentum Density
4. $T^{00} \Rightarrow$ Momentum Flux

$$\nabla_\mu T^{\mu\nu} = 0$$

Dust:

$$\boxed{T^{\mu\nu} = \rho u^\mu u^\nu}$$

Perfect Fluid:

$$\boxed{T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P\eta^{\mu\nu}}$$

3.7.1 Equation of State

$$\boxed{P = w\rho}$$

1. $w = \frac{1}{3} \Rightarrow$ Radiation
2. $w = 0 \Rightarrow$ Dust
3. $w = -\frac{1}{3} \Rightarrow$ Cosmic Strings
4. $w = -\frac{2}{3} \Rightarrow$ Domain Walls
5. $w = -1 \Rightarrow$ Cosmological Constant

3.7.2 EM Stress-Energy Tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}(F^{\mu\alpha}F^{\nu}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F^{\alpha\beta})$$
$$T^{\mu}{}_{\mu} = \eta_{\mu\nu}T^{\mu\nu} = 0$$
$$T^{0i} = (\vec{S})^i = \left(\frac{\vec{E} \times \vec{B}}{\mu_0}\right)^i$$

4 General Relativity

4.1 Gravitational Redshift

$$\frac{\nu'}{\nu} = \frac{E'}{E} = 1 - \frac{\triangle\phi}{c^2}$$

For a region of constant G field,

$$\frac{\nu'}{\nu} = 1 - \frac{gy}{c^2}$$

$$\Box\phi = -\frac{4\pi G}{c^2}\rho_E,\ \rho_E = \text{Energy Density}$$

4.2 Geodesic Equations

$$u^\mu \nabla_\mu u^\nu = 0,\ \frac{d}{d\tau} = u^\mu \partial_\mu$$
$$\frac{d}{d\tau}(u_\rho) = \frac{1}{2}(\partial_\rho g_{\mu\nu})u^\mu u^\nu = \frac{1}{2}g_{\mu\nu,\rho}u^\mu u^\nu$$

For *massless* particle,

$$\frac{d}{d\lambda}(p_\rho) = \frac{1}{2}(\partial_\rho g_{\mu\nu})p^\mu p^\nu$$
$$\frac{d^2}{d\tau^2}x^\rho = -\Gamma^\rho{}_{\mu\nu}u^\mu u^\nu$$

4.3 Christoffel Symbol

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

4.4 Covariant Derivatives

$$\nabla_\mu \phi = \partial_\mu \phi$$
$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta}V^\beta$$
$$\nabla_\mu V_\alpha = \partial_\mu V_\alpha - \Gamma^\beta_{\mu\alpha}V_\beta$$
$$\nabla_\mu T^{\alpha\beta} = \partial_{mu}T^{\alpha\beta} + \Gamma^\alpha_{\mu\rho}T^{\rho\beta} + \Gamma^\beta_{\mu\rho}T^{\alpha\rho}$$
$$\nabla_\mu T_{\alpha\beta} = \partial_{mu}T_{\alpha\beta} - \Gamma^\rho_{\mu\alpha}T_{\rho\beta} - \Gamma^\rho_{\mu\beta}T_{\alpha\rho}$$

4.5 Riemann Curvature Tensor

4.6 Einstein's Field Equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$
$$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

4.7 Blackhole Metrics

Vacuum solutions,

$$\Rightarrow G^{\mu\nu} = 0 \;\Rightarrow R^{\mu\nu} = 0 \;\Rightarrow R = 0$$

Schwarzschild Blackholes

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{(1 - \frac{2GM}{r})} + r^2 d\Omega^2$$
$$\text{where, } \; d\Omega = d\theta^2 + \sin^2(\theta)d\phi^2$$
$$r_s = \frac{2GM}{r}, \text{ is the } \textit{Event Horizon}$$

Reissner-Nordström Blackholes

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2$$
$$\text{where, } \; f(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r}$$
$$\text{This has 2 `Horizon' solutions,}$$
$$\text{Event Horizon: } r_+ = \frac{1}{2}(r_s + \sqrt{r_s^2 - 4r_Q^2})$$
$$\text{Cauchy Horizon: } r_- = \frac{1}{2}(r_s - \sqrt{r_s^2 - 4r_Q^2})$$
$$r_s = 2r_Q \Rightarrow \text{Extremal Solution}$$

Kerr Blackholes

5 Cosmology

5.1 Friedman-Robertson-Walker

The Metric of the Universe

$$ds^2 = -dt^2 + a(t)^2(\frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2 d\Omega^2)$$

$$k = -1 \Rightarrow \text{Open Universe}$$
$$k = 0 \Rightarrow \text{Flat Universe}$$
$$k = 1 \Rightarrow \text{Closed Universe}$$

5.2 Friedman Equations

First Friedman Equation

$$\left(\frac{a'(t)}{a(t)}\right)^2 = H(t)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a(t)^2}$$

Expansion velocity

$$v = H(t)l = \left(\frac{a'(t)}{a(t)}\right)l$$

Second Friedman Equation

$$\frac{a''(t)}{a(t)} = -\frac{4\pi G}{3}(\rho + 3P)$$

Equation of State

$$P = w\rho$$

Various equations of state

$$w = \frac{1}{3} \rightarrow \text{Radiation} \Rightarrow \rho \propto a^{-4}$$
$$w = 0 \rightarrow \text{Dust} \Rightarrow \rho \propto a^{-3}$$
$$w = -\frac{1}{3} \rightarrow \text{Cosmic String} \Rightarrow \rho \propto a^{-2}$$
$$w = -\frac{2}{3} \rightarrow \text{Domain Walls} \Rightarrow \rho \propto a^{-1}$$
$$w = -1 \rightarrow \text{Cosmological Constant}$$
$$\Rightarrow \rho \propto a^0 = 1$$

5.3 Cosmological Redshift

6 Useful Identities

$$\eta^{\rho\nu}\eta_{\nu\mu} = \delta^\rho{}_\mu$$
$$\nabla_\mu T^{\mu\nu} = \nabla_\nu T^{\mu\nu} = 0$$
$$g^{\mu\nu}g_{\nu\mu} = g^\mu{}_\mu = 4$$
$$\nabla_\rho g_{\mu\nu} = 0$$
$$\frac{d}{d\tau}(u_\rho u^\rho) = 0$$