7270 Quantum Field Theory

University of Colorado, Boulder (Fall 2020)

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The textbook of relevance for this course is “Quantum Field Theory” by Mark Srednicki. It will only be followed very loosely however, for which these notes will elaborate on the differences.

All notes were taken real-time in the class (i.e. there are bound to be typos) taught by Professor Paul Romatschke. For information on this class, refer to the canvas course page.

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Contents

Chapter 1

Introduction

§1.1 Units and Notation

Before diving in to the meat of the subject, it is necessary to establish the system of units adopted and some important notation that will follow us through the course. The unit system used in the class is referred to *natural units*, in which $c = \hbar = k_B = 1$. This mean that, the only effective unit left is energy, for which all other units can be expressed in units of energy.

Since this course contains a treatment of relativistic quantum mechanics, it is necessary to distinguish between 3-vectors (with spatial components) and 4-vectors (with spacetime components). To do so, Latin indices $(i, j, k, \ldots)$ are used to denote 3-vectors while Greek indices $(\mu, \nu, \rho, \ldots)$ are used for 4-vectors. Furthermore, Einstein sum notation is employed in which a sum over repeated indices is applied. For instance,

$$A^\mu A_\mu = A^\mu g_{\mu\nu} A^\nu = A_0 A^0 + A_1 A^1 + A_2 A^2 + A_3 A^3, \tag{1.1}$$

where the $g_{\mu\nu}$ is the metric tensor. As for derivatives, we use the shorthand notation

$$\begin{aligned} \boldsymbol{\nabla}^2 &= \partial_i \partial_i \\ \text{where} \quad \partial_i &= \frac{\partial}{\partial x_i}, \end{aligned} \tag{1.2}$$

which translates similarly for derivatives with Greek indices. We will mostly be using the metric tensor for Minkowski (flat) spacetime in this course, in which the mostly positive sign convention is adopted,

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.3}$$

Note that this is **not** the same convention as adopted in the recommended text. Finally if integrals and sums are written without any limits, they are taken to be integrals/sums over the entire domain

$$\int dx = \int_{-\infty}^{\infty} dx, \quad \sum = \sum_{j=-\infty}^{\infty}. \tag{1.4}$$

§1.2 The Quantum Partition Function

This section provides a brief discussion of the partition function in the context of quantum mechanics by analogy of the time evolution operator and the density operator. A good reference for this section is provided in Chp. 1 of "Basics of Thermal Field Theory" by Laine and Vuorinen. To start off, we begin with the non-relativistic, 1-dimensional Schrödinger's equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[-\frac{\partial_x^2}{2m} + \hat{V}(x) \right] \psi(x,t). \tag{1.5}$$

By utilizing a separation of variables, we get the energy eigen equation

$$\hat{H}\psi(x) = E\psi(x), \tag{1.6}$$

which when solved, can be used to construct the separable solution

$$\psi(x,t) = e^{-i\hat{H}t}\psi(x). \tag{1.7}$$

In thermodynamics, an important quantity is the *partition function* $Z(T)$. Having this allows us to compute other quantities of interest such as the free energy F, and entropy S, by the relations

$$F = -T \ln Z, \quad S = -\frac{\partial F}{\partial T}. \tag{1.8}$$

In statistical mechanics, it is customary to define $\beta = 1/T$, which then grants the definition of the partition function as

$$\boxed{Z(T) = \mathrm{Tr}\left[e^{-\beta \hat{H}} \right]}, \tag{1.9}$$

where in quantum mechanics, the trace is over the entire Hilbert space. In quantum mechanics, given that we have solved for the basis the energy eigenstates $|n\rangle$ (s.t. $\hat{H}|n\rangle = E_n |n\rangle$), we can use the completeness of this basis to compute the partition function

$$\begin{aligned} Z(T) &= \sum_{n=0}^{\infty} \langle n| e^{-\beta \hat{H}} |n\rangle \\ \Rightarrow \quad &\boxed{Z(T) = \sum_{n=0}^{\infty} e^{-\beta E_n}}. \end{aligned} \tag{1.10}$$

These exponents in the sum are known as *Boltzmann factors*. Let us now consider a simple example.

Example:
Consider the quantum harmonic oscillator, in which the Hamiltonian is given by

$$\hat{H} = -\frac{\partial_x^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}, \tag{1.11}$$

and has energy eigenvalues

$$E_n = \omega\left(n + \frac{1}{2}\right). \tag{1.12}$$

This results in the partition function

$$\begin{aligned} Z(T) &= e^{-\beta\omega/2}\sum_{n=0}^{\infty} e^{-\beta\omega n} \\ &= \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}} \\ &= \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}. \end{aligned} \tag{1.13}$$

§1.3 The Path Integral

In this section, we are going to see how the quantum partition function leads to a derivation of the path integral formulation of quantum mechanics. The more common Schrödinger picture answers the question of the possible positions and their probabilities for a particle to end up, starting off in some position. The path integral formulation on the other hand, tells us the probability of transitioning from one position to another by consider all possible intermediate paths between these 2 points. Since the partition function serves to scope out and in some sense, collate the entire state space, it seems the appropriate tool to develop such a formulation.

To start off, let us consider the partition function in the position basis. The position basis is going to be an infinite dimensional Hilbert space, and so will have integrals replacing sums of the basis states. As such, we have that the partition function can be written as

$$Z(T) = \int dx \, \langle x| \, e^{-\beta\hat{H}} \, |x\rangle \, . \tag{1.14}$$

At this point, we realize that since β is simply a constant (not an operator), it will always commute with $\hat{H}$. Furthermore, any factorization of β will also commute with $\hat{H}$ and so one can write

$$e^{-\beta\hat{H}} = e^{-\epsilon\hat{H}}e^{-\epsilon\hat{H}}e^{-\epsilon\hat{H}}\dots e^{-\epsilon\hat{H}}, \tag{1.15}$$

where $\epsilon = \beta/N$ for some positive integer N. This splitting of the Boltzmann factor allows us to insert unities between these different products, which by the theorem of completeness, can take different basis forms

$$\mathbf{1} = \int dx \, |x\rangle \, \langle x| = \int \frac{dp}{2\pi} \, |p\rangle \, \langle p| = \dots \tag{1.16}$$

Notice that there is a difference of 2π that arises between these position and momentum basis, which comes from normalization ($\langle x|p\rangle = e^{ipx}$). To keep track of these inserted unities, we can simply index the variables, starting from the rightmost splitting of the Boltzmann factor. That is consider the insertion of 3 unities, 2 of which are position bases and the other a momentum basis. These are inserted sequentially as follows.

$$\begin{aligned} Z(T) &= \int dx \, \langle x| \, e^{-\epsilon \hat{H}} \ldots e^{-\epsilon \hat{H}} \, |x\rangle \\ &= \int dx dx_1 \, \langle x| \, e^{-\epsilon \hat{H}} \ldots e^{-\epsilon \hat{H}} \, |x_1\rangle \, \langle x_1|x\rangle \\ &= \frac{1}{2\pi} \int dx dx_1 dp_1 \, \langle x| \, e^{-\epsilon \hat{H}} \ldots e^{-\epsilon \hat{H}} \, |p_1\rangle \, \langle p_1| \, e^{-\epsilon \hat{H}} \, |x_1\rangle \, \langle x_1|x\rangle \\ &= \frac{1}{2\pi} \int dx dx_1 dp_1 dx_2 \, \langle x| \, e^{-\epsilon \hat{H}} \ldots e^{-\epsilon \hat{H}} \, |x_2\rangle \, \langle x_2|p_1\rangle \, \langle p_1| \, e^{-\epsilon \hat{H}} \, |x_1\rangle \, \langle x_1|x\rangle \, . \end{aligned} \tag{1.17}$$

If this process of inserting identities is continued recursively for N position and momentum bases, we end up with $2N+1$ integrals with differential element

$$\frac{dx \, (dx_1 dx_2 \ldots dx_N) \, (dp_1 dp_2 \ldots dp_N)}{(2\pi)^N}. \tag{1.18}$$

The integrand would be constituted by matrix elements of the type

$$\langle x_{j+1}|p_j\rangle \, \langle p_j| \, e^{-\epsilon \hat{H}} \, |x_j\rangle = e^{ip_j x_{j+1}} \, \langle p_j| \, e^{-\epsilon \hat{H}} \, |x_j\rangle \tag{1.19}$$

$$= e^{ip_j x_{j+1}} \, \langle p_j| \, e^{-\epsilon \left[\frac{p_j^2}{2m} + V(x_j)\right] + \mathcal{O}(\epsilon^2)} \, |x_j\rangle \tag{1.20}$$

$$\approx e^{ip_j x_{j+1}} e^{-\epsilon \left[\frac{p_j^2}{2m} + V(x_j)\right]} \, \langle p_j|x_j\rangle \tag{1.21}$$

$$= e^{ip_j x_{j+1}} e^{-\epsilon \left[\frac{p_j^2}{2m} + V(x_j)\right]} e^{-ip_j x_j}, \tag{1.22}$$

if $N \gg 1$, so that higher-order corrections due to non-commutation of the momentum and potential operators can be dropped. As such, we are left with the integrand matrix elements

$$\langle x_{j+1}|p_j\rangle \, \langle p_j| \, e^{-\epsilon \hat{H}} \, |x_j\rangle \approx e^{-\epsilon \left[\frac{p_j^2}{2m} + V(x_j) - p_j \frac{x_{j+1} - x_j}{\epsilon}\right]} \tag{1.23}$$

$$\Rightarrow \quad Z(T) = \lim_{N \to \infty} \int dx \left[\prod_{k=1}^{N} \frac{dx_k dp_k}{2\pi}\right] e^{-\epsilon \sum_{j=0}^{N} \left[\frac{p_j^2}{2m} + V(x_j) - p_j \frac{x_{j+1} - x_j}{\epsilon}\right]} \, \langle x_1|x\rangle \, , \tag{1.24}$$

where the $\langle x|$ on the very left of the integrand is identified as $\langle x_{N+1}|$. This implies that we have asserted a periodic boundary condition of $x_{N+1} = x_1$. Since eigenstates are orthonormal, this implies that $\langle x_1|x\rangle = \delta(x_1 - x)$, which simplifies the partition function to

$$Z(T) = \lim_{N \to \infty} \int \left[\prod_{k=1}^{N} \frac{dx_k dp_k}{2\pi}\right] e^{-\epsilon \sum_{j=0}^{N} \left[\frac{p_j^2}{2m} + V(x_j) - p_j \frac{x_{j+1} - x_j}{\epsilon}\right]}. \tag{1.25}$$

§1.3.1 Imaginary Time

It is useful to think of the N discretized spatial points as points on a line, separated by $\epsilon = \beta/N$ segments. To this end, we define a new discrete variable $\tau \equiv j\epsilon = j\beta/N$. The periodic boundary condition on space then imposes that the line forms a closed loop, which we call the *thermal circle* of circumference β ($x(\tau = \beta) = x(\tau = 0)$) as visualized in Fig. 1.1.

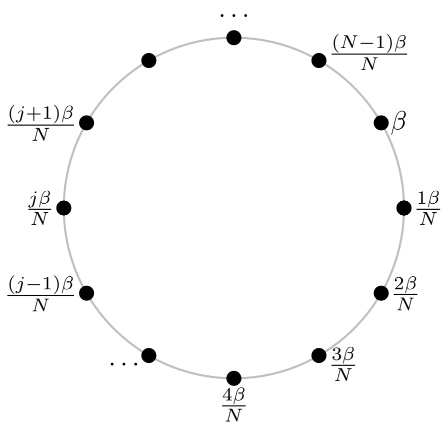


Figure 1.1: Thermal circle with spacings β/N.

Of course in the limit as $N \to \infty$, τ serves as a continuous variable which we will refer to as *imaginary time* (a.k.a Euclidean time) for which the names will become apparent soon. In this continuous variable limit, the partition function in Eq. (1.25) will be modified by replacing

$$\lim_{N\to\infty} \frac{x_{j+1} - x_j}{\epsilon} \quad \to \quad \frac{dx(\tau)}{d\tau}, \tag{1.26a}$$

$$\lim_{N\to\infty} \epsilon \sum_{j=0}^{N} \quad \to \quad \int_0^\beta d\tau, \tag{1.26b}$$

$$\lim_{N\to\infty} \left[\prod_{k=1}^{N} \frac{dx_k dp_k}{2\pi} \right] \quad \to \quad \frac{\mathcal{D}_x \mathcal{D}_p}{2\pi}, \tag{1.26c}$$

rendering the partition function to be written as

$$\boxed{Z(T) = \int \frac{\mathcal{D}_x \mathcal{D}_p}{2\pi} \exp\left[-\int_0^\beta d\tau \left(\frac{p^2(\tau)}{2m} + V(x(\tau)) - p(\tau)\frac{dx(\tau)}{d\tau} \right) \right]}. \tag{1.27}$$

Eq. (1.27) above is referred to as the *continuum path integral*, which is a path integral over the 2 functions $x(\tau)$ and $p(\tau)$. We can in fact simplify this expression by noticing that $V(x)$ is **only** a function of $x(\tau)$ and not $p(\tau)$. So we can first "integrate out" all the $p(\tau)$ variables by going back to discrete space and evaluating

$$\begin{aligned}\int \frac{dp_j}{2\pi} e^{-\epsilon\left[\frac{p_j^2}{2m} - p_j \frac{x_{j+1}-x_j}{\epsilon}\right]} &= \sqrt{\frac{m}{2\pi\epsilon}} e^{-\frac{m(x_{j+1}-x_j)^2}{2\epsilon}} \\ \Rightarrow \quad Z(T) = \lim_{N\to\infty} \int \left[\prod_{k=1}^{N} \frac{dx_k}{\sqrt{2\pi\epsilon/m}}\right] & e^{-\epsilon \sum_{j=0}^{N}\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 + V(x_j)\right]}.\end{aligned} \tag{1.28}$$

This can then be expressed in the continuum limit as

$$\begin{aligned} Z(\beta) &= C \int_{x(0)=x(\beta)} \mathcal{D}_x \exp\left[-\int_0^\beta d\tau \left(\frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau))\right)\right], \\ \text{where} \quad C &= \left(\frac{m}{2\pi}\epsilon\right)^{N/2}. \end{aligned} \tag{1.29}$$

The factor C appears to diverge in the limit of $N \to \infty$, however since it seems to play no role in the actual dynamics of the system, we will do the physicists trick of sweeping weird mathematical anomalies under the rug. And that was all she wrote. At least for the time being.

The integrand in the exponent however should look familiar to a physicist's eye, for which (despite a sign discrepancy) we term it the *Euclidean Lagrangian* (since it is parameterized by Euclidean time):

$$L_E = \frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau)). \tag{1.30}$$

It is only natural then that the integral be referred to as the *Euclidean action*

$$S_E = \int_0^\beta L_E. \tag{1.31}$$

The sign discrepancy alluded to earlier can be somewhat fixed if we made the transformation $\tau = it$ referred to as a *Wick rotation* (a $\pi/2$ rotation in the complex plane), so that we instead have

$$L_E = -\frac{m}{2}\left(\frac{dx(t)}{dt}\right)^2 + V(x). \tag{1.32}$$

This then grants us that $L_E = -L(t = -i\tau)$, where L is the Lagrangian we are familiar to from classical mechanics. This transformation is the reason why τ is also referred to as Euclidean time in the context of relativistic QFTs, since in the Minkowski metric we have the invariant spacetime interval written as

$$ds^2 = d\boldsymbol{x}^2 - dt^2, \tag{1.33}$$

the Wick rotation would grant us the invariant interval in terms of imaginary time as

$$ds^2 = d\boldsymbol{x}^2 + d\tau^2, \tag{1.34}$$

which is just a length in ordinary Euclidean space! Correspondingly, we have that for the action,

$$e^{-S_E} = e^{iS}, \tag{1.35}$$

where S is the classical action. It is not surprising that we see imaginary numbers cropping up, since we are indeed dealing with the quantum mechanical partition function.

§1.3.2 Solving the Path Integral

Having now this new means of computing the quantum mechanics partition function, it is only useful if we know how to solve it. To do so, we first consider again the example of the quantum harmonic oscillator. This will allows us to compare our result with that obtained in Eq. (1.13).

The first step would be to write down the Euclidean action for this system

$$S_E = \int_0^\beta d\tau \left[\frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + \frac{m\omega^2 x(\tau)^2}{2}\right]. \tag{1.36}$$

The quadratic form of the integrand in the action makes moving to Fourier space an appealing next step, for which we shall oblige. The discrete Fourier transform (due to the periodic boundary condition on $x(\tau)$) is written as

$$x(\tau) = T\sum_n e^{i\omega_n\tau} x_n \tag{1.37}$$

where $\omega_n = 2\pi nT$ are known as the *Matsubara frequencies* and x_n are the Fourier coefficients. The temperature prefactor T, here is inserted for convenience and is purely a convention. We know that $x(\tau)$ is a real position coordinate, so it must be that $x_n^* = x_{-n}$, for which we can expand these coefficients as $x_n = a_n + ib_n$. The general quadratic form in an integral can be evaluated using the Fourier transform as

$$\begin{aligned}\int_0^\beta d\tau x(\tau)y(\tau) &= T^2\sum_{n,m} x_n y_m \int_0^\beta d\tau e^{i\tau(\omega_n+\omega_m)} \\ &= T^2\beta\sum_{n,m} x_n y_m \delta_{n,-m} \\ &= T\sum_n x_n y_{-n} = T\sum_n x_n y_n^*.\end{aligned} \tag{1.38}$$

Using this identity, the action for the harmonic oscillator can then be evaluated as

$$\begin{aligned} S_E &= \int_0^\beta d\tau \left[\frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + \frac{m\omega^2 x(\tau)^2}{2} \right] \\ &= \frac{mT}{2}\sum_n x_n \left[i\omega_n i\omega_{-n} + \omega^2 \right] x_{-n} \\ &= \frac{mT}{2}\sum_n \left[\omega^2 - \omega_n\omega_{-n} \right] (a_n^2 + b_n^2) \\ &= \frac{mT}{2}\sum_n \left[\omega^2 + \omega_n^2 \right] (a_n^2 + b_n^2) \\ &= \frac{mT\omega^2}{2} a_0^2 + mT \sum_n \sum_{n=1}^{\infty} \left[\omega^2 + \omega_n^2 \right] (a_n^2 + b_n^2), \end{aligned} \tag{1.39}$$

where we used the fact that ω_n is linear in n, $b_0 = 0$ and the sum over n without the $n = 0$ is perfectly symmetric. Since we are now in Fourier space, the path integral must also be evaluated in Fourier space for which the Jacobian between $x(\tau)$ and the x_n (and thus a_n, b_n) coefficient must be employed. This is given by

$$C\mathcal{D}_x = C\left|\det\left[\frac{\delta x(\tau)}{\delta x_n}\right]\right| da_0 \prod_{n=1}^{\infty} da_n db_n. \tag{1.40}$$

We simplify this notation by absorbing the jacobian into the constant C such that

$$C' = C\left|\det\left[\frac{\delta x(\tau)}{\delta x_n}\right]\right|. \tag{1.41}$$

As such, we get the path integral

$$\begin{aligned} Z &= C'\int da_0 \int \left[\prod_{n=1}^{\infty} da_n db_n\right] \exp\left[-\frac{mT\omega^2}{2} a_0^2 - mT\sum_n \sum_{n=1}^{\infty} \left[\omega^2 + \omega_n^2 \right] (a_n^2 + b_n^2) \right] \\ &= C'\sqrt{\frac{2\pi}{mT\omega^2}} \prod_{n=1}^{\infty} \frac{\pi}{mT(\omega_n^2 + \omega^2)}. \end{aligned} \tag{1.42}$$

However, the constant C' still needs to be evaluated, for which one can employ the *effective field theory matching* method which leads to the result

$$C' = \frac{T}{2\pi}\sqrt{2\pi m T} \prod_{n=1}^{\infty} \frac{mT\omega_n^2}{\pi}. \tag{1.43}$$

Plugging this back into the path integral formula, we get

$$\begin{aligned} Z(T) &= \frac{T}{\omega} \prod_{n=1}^{\infty} \frac{\omega_n^2}{\omega^2 + \omega_n^2} \\ &= \frac{T}{\omega} \prod_{n=1}^{\infty} \frac{1}{1 + \frac{\omega^2}{(2\pi n T)^2}} = \frac{1}{2\sinh\left(\frac{\omega}{2T}\right)}, \end{aligned} \tag{1.44}$$

which indeed matches the result in Eq. (1.13). The final expression arose from the use of the identity

$$\frac{\sinh(x)}{x} = \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{\pi^2 n^2}\right). \tag{1.45}$$

§1.3.3 Path Integral Formulation of Quantum Mechanics

We shall look more into the interpretation of quantum mechanics in terms of the path integral we have just derived. In this path integral formulation, we no longer work with operators, eigenvalues and wavefunctions. Instead, all the elements required to set-up the observables are purely classical, while the heavy lifting is provided by the infinite number of path integrals in this representation. So far, the path integral we have seen seems to have a lot more to do with statistical physics rather than quantum field theory. This section will draw a clear link between quantum field theory and the path integral.

To begin, we consider again the transition matrix element from position x' at time t', to position x'' at time t''

$$\langle x'', t'' | x, t\rangle = \langle x''|\, e^{-i\hat{H}(t''-t')}\, |x'\rangle\,. \tag{1.46}$$

From here, we perform a similar derivation of the path integral, but instead using the integral $i\int_{t'}^{t''}$ instead of $\int_0^\beta d\tau$ (i.e. by transforming $\tau \to it$) and positions taken to be $x(\beta) \to x'' = x(t'')$, $x(0) \to x' = x(t')$. This leads to the path integral

$$\boxed{\langle x''|\, e^{-i\hat{H}(t''-t')}\, |x'\rangle = C\int \mathcal{D}_x e^{i\int_{t'}^{t''} dtL}}, \tag{1.47}$$

where L is the classical Lagrangian of a point particle. This brings us to the conclusion that the quantum mechanical transition amplitude and the quantum mechanical partition function are analogous, which are compared below:

Transition Amplitude	**Partition Function**
Provides probability information	Provides thermodynamic equilibrium information
Uses real-time, t	Uses temperature parameter, τ
Formulated in Minkowski space	Formulated in Euclidean space
Path integral is ill-defined (e^{iS})	Path integral is well-defined (e^{-S_E})

Table 1.1: Analogies between the quantum mechanical transition amplitude and the quantum mechanical partition function.

It is useful to note that since the partition function consists of a well-defined integral, it can be evaluated via numerical Monte Carlo methods which opens the door to numerical solutions.

At this point however, the path integral formulated in Eq. (1.47) still only addresses traditional quantum mechanics which quantizes classical point particles. How then do we transition from quantum mechanics to quantum field theory? Well, we will have to trade-in the classical Lagrangian $L(x)$ for point particles into the Lagrangian density, $\mathcal{L}(\phi)$ for classical fields ϕ. As such, the action would now be $S = \int dtd^3x\mathcal{L}$, and so the path integral become

$$Z = \int \mathcal{D}_x e^{-\int d\tau L_E(x)} \quad \rightarrow \quad Z = \int \mathcal{D}_\phi e^{-\int d\tau d^3 \mathcal{L}_E(\phi)}, \tag{1.48}$$

where the path integral now runs over all field configurations. It is now time to get into field theory.

Chapter 2

Introductory Field Theory

This chapter will first touch on classical field theory, which we will then quantize to grant us a quantum field theory. This will eventually allow us to construct a relativistic theory of quantum mechanics, i.e. a formalism that combines special relativity and quantum mechanics into a single theory. However, before we get started, it would be useful to have a review of classical Lagrangian mechanics for point particles, before we get into the field theoretic descriptions.

§2.1 Review of Lagrangian Mechanics

Starting with the classical action for a one-degree of freedom system,

$$S = \int_{t_i}^{t_f} dt L\left(q, \dot{q}\right), \tag{2.1}$$

we consider the derivative of the action with respect to $q(t)$, written as

$$\begin{aligned}
\frac{\delta S[q(t)]}{\delta q(t')} &= \lim_{\varepsilon \to 0} \frac{\int_{t_i}^{t_f} dt L\left(q(t) + \varepsilon\delta(t-t'), \dot{q}(t) + \varepsilon\dot{\delta}(t-t')\right) - \int_{t_i}^{t_f} dt L\left(q(t), \dot{q}(t)\right)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\int_{t_i}^{t_f} dt \left[\frac{\partial L}{\partial q(t)}\varepsilon\delta(t-t') + \frac{\partial L}{\partial \dot{q}(t)}\varepsilon\dot{\delta}(t-t')\right]}{\varepsilon} \\
&= \frac{\partial L}{\partial q(t')} + \frac{\partial L}{\partial \dot{q}(t)}\delta(t-t')\bigg|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)}\delta(t-t') \\
&= \frac{\partial L}{\partial q(t')} - \frac{d}{dt'}\frac{\partial L}{\partial \dot{q}(t')}.
\end{aligned} \tag{2.2}$$

To extremize S then, we set the first derivative to zero which grants us the Euler-Lagrange equation

$$\frac{\partial L}{\partial q(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}(t)} = 0. \tag{2.3}$$

It is then easy to generalize this result to a system with multiple degrees of freedom by adding indices to the generalized coordinates, which gives us

$$\frac{\partial L}{\partial q_j(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j(t)} = 0. \tag{2.4}$$

This mechanics for point particles tell us about the dynamics of localized objects in space. On the other hand, classical field theory tells us about the dynamics of *fields* which are objects that extend over all space (e.g. temperature scalar field $T(\boldsymbol{x}, t)$, wind vector field $\boldsymbol{v}(\boldsymbol{x}, t)$). Additionally, we are going to explore relativistic field theories, which **must** conform to the specific symmetries in special relativity (i.e. Lorentz invariance).

§2.2 Classical Field Theory

More specifically, a relativistic field theory is on in which its classical action is invariant under the group of Lorentz transformations. Recall that in special relativity, space and time are no long independent, and are together known as spacetime. As such, our classical definition of the action $S = \int dtL$, will no longer allow the preservation of Lorentz symmetries, and must be modified such that it does. To do so, we introduce a new object known as the *Lagrangian density*, $\mathcal{L}$, such that

$$S = \int d^4x\mathcal{L}, \tag{2.5}$$

where $d^4x = dtd^3x$ is the spacetime volume element. The next question we should answer is, what kind of field should our theory consist of. Arguably, the simplest case would be to start with a scalar field which denote by $\phi(x)$, corresponding to a single degree of freedom at every spacetime point x_μ. From this, the Lagranian density can then be constructed in terms of the field and its derivatives, only limited by the fact that the Lagrangian density itself must be a scalar (and consequently Lorentz invariant).

However for the time being, we will consider Lagrangian densities of the form

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi), \tag{2.6}$$

where $V(\phi)$ is an arbitrary well-behaved function of ϕ. The resulting action is then

$$S = -\int d^4x \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + V(\phi)\right], \tag{2.7}$$

which when extremized produces the equation of motion

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0}. \tag{2.8}$$

We can also derive a Euclidean Lagrangian density by making the change of variable $t \to -i\tau$, which gives

$$\mathcal{L}_E = \frac{1}{2}\partial_\tau\phi\partial_\tau\phi + \frac{1}{2}\partial_j\phi\partial_j\phi + V(\phi) \tag{2.9}$$

$$\Rightarrow \quad S_E = \int d^4x_E \left[\frac{1}{2}\partial_a\phi\partial_a\phi + V(\phi)\right], \tag{2.10}$$

where $d^4x_E = d\tau d^3x$ and $a = 1, 2, 3, 4$ is the 4-dimensional Euclidean index.

> **Note**: Often, the subscript "E" on x_E will be dropped for lighter notation when the fact that we are working in Euclidean coordinates is unambiguous. Furthermore, we will simply use x to denote the 4-vector of coordinates, while boldfaced $\boldsymbol{x}$ is reserved for the 3-space coordinate vector.

The geometry of this 4-dimensional Euclidean volume is called a *thermal cylinder* since τ forms a thermal circle ($\phi(\tau = 0) = \phi(\tau = \beta)$), from which the other spatial elements dimensions extend this circle into a cylinder.

> **Note**: Henceforth, the terms "Lagrangian" and "Lagrangian density" will be used interchangeably unless a need arises to differentiate the 2 is necessary. Otherwise, the Lorentz invariant object in assumed.

§2.3 Relativistic Quantum Field Theory

As in the path integral for point particles, quantum nature of the path integral for quantum fields would be encoded in the infinite-dimensional path integral measure $\mathcal{D}_\phi$. As such, knowledge of the classical action allows us full access to the quantum field theoretic partition function. We shall refer to every physical system comprised of a unique Lagrangian, a *quantum field theory* (QFT). Unfortunately, not all classical actions give rise to sensible QFTs and neither can all QFTs be solved. There nonetheless do exist solvable QFTs, so we shall focus on those.

§2.3.1 Free Scalar Field Theory

The first solvable quantum field theory we shall study is known as the *free scalar field theory.* In this theory, the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2\phi^2}{2}, \tag{2.11}$$

is constructed by considering the continuum limit of an infinite number of coupled harmonic oscillators, where the potential is quadratic in ϕ. With such a Lagrangian, we can derive the equations of motion by means of the Euler-Lagrange equations which for scalar fields was given in Eq. (2.8). Plugging in the Lagrangian then gives the equation of motion

$$\left(\Box - m^2\right)\phi = 0, \tag{2.12}$$

where $\Box = \boldsymbol{\nabla}^2 - \partial_0^2$ is the *d'Alembert operator* and the equation of motion above is known as the *Klein-Gordon* equation.

Alternatively, since the potential is quadratic, we can employ the use of Fourier transforms to solve for the partition function, similar to what we did for the point particle quantum harmonic oscillator, but now also having to Fourier transform all spatial dimensions. To do so, we first work in generality and take that there are D spatial dimensions (instead of assuming 3). We will also first work in a box of edge length $x^j \in [-L/2, L/2]$, then later take the limit as $L \to \infty$.With this, we take the Fourier transform as

$$\phi(\tau, \boldsymbol{x}) = \frac{T}{L^D} \sum_n \sum_{k_1, k_2, \ldots, k_D} \tilde{\phi}(\omega_n, \boldsymbol{k}) \, e^{i\omega_n \tau + i\boldsymbol{K} \cdot x}, \tag{2.13}$$

where again $\omega_n = 2\pi n T$ are the Matsubara frequencies and k_j is discretized with steps $\Delta k = 2\pi/L$. Plugging this into the Euclidean action gives as

$$S_E = \frac{T}{2L^D} \sum_{\omega_n, \boldsymbol{k}} \left(\omega_n^2 + \boldsymbol{k}^2 + m^2\right) \left|\tilde{\phi}(\omega_n, \boldsymbol{k})\right|^2, \tag{2.14}$$

$$\Rightarrow \quad e^{-S_E} = \prod_{\boldsymbol{k}} \exp\left[-\frac{T}{2L^D} \sum_{\omega_n} \left(\omega_n^2 + \boldsymbol{k}^2 + m^2\right) \left|\tilde{\phi}(\omega_n, \boldsymbol{k})\right|^2\right]. \tag{2.15}$$

Notice that the exponent in each product term above is exactly the same as that for the point particle quantum harmonic oscillator in Eq. (1.39), just with different coefficients. To restate the harmonic oscillator's action, it was

$$S_E^{(\mathrm{HO})} = \frac{m_{(\mathrm{HO})} T}{2} \sum_n \left[\omega_{(\mathrm{HO})}^2 + \omega_n^2\right] |x_n|^2. \tag{2.16}$$

But for the free scalar field, we instead have the replacements

$$m_{(\mathrm{HO})} \to \frac{1}{L^D} \tag{2.17a}$$

$$\omega_{(\mathrm{HO})}^2 \to \boldsymbol{k}^2 + m^2 \equiv E_k^2, \tag{2.17b}$$

which gives the result

$$Z_{\mathrm{free}} = \prod_{\boldsymbol{k}} \frac{1}{2\sinh\left(\frac{E_k \beta}{2}\right)} = \exp\left[-\sum_{\boldsymbol{k}} \frac{E_k \beta}{2} + \ln\left(1 - e^{-\beta E_k}\right)\right]. \tag{2.18}$$

Now, we take the limit as $L \to \infty$, which changes the sums to integrals

$$\begin{gathered} \lim_{L \to \infty} \frac{1}{L^D} \sum_{\boldsymbol{k}} \quad \to \quad \int \frac{d^D \boldsymbol{k}}{(2\pi)^D} \\ \Rightarrow \quad \boxed{\ln Z_{\mathrm{free}} = -\frac{V}{T} \int \frac{d^D \boldsymbol{k}}{(2\pi)^D} \left[\frac{E_k}{2} + T \ln\left(1 - e^{-\beta E_k}\right)\right]}, \end{gathered} \tag{2.19}$$

where $V = L^D$ is the volume of D-dimensional space. The problem however, is that although we have a nice analytic form, the integral above is badly divergent (singular) for any finite temperature. This will actually be a recurring issue in all of QFT, for which useful physics can actually still be extracted, but is usually "hidden" under these divergences. The procedure to extract such physics consist *regularization* (identifying and subtracting away divergences) and *renormalization* (modification of the theory on physical grounds). These will appear to be rather adhoc in their construction, but are necessary for a consistent theory.

§2.3.2 Regularization

Broadly speaking, regularization is a procedure in which quantities which have singularities are modified so as to make them finite. This is usually done by introducing suitable parameters known as *regulators*. It is best to understand this through an application, for which we will consider the free scalar field partition function.

Going back to the free scalar field, we first notice that in the zero temperature limit, the free energy function, $F = -T \ln Z$, becomes just a sum over all zero-point harmonic oscillator energies.

$$F = \sum_{\boldsymbol{k}} \frac{E_k}{2}. \tag{2.20}$$

Since in the continuum limit, there is an infinite number of oscillators with finite zero-point energy, the result is divergent. Zero-point energies are not new to us, so we are going to attempt regularizing this (i.e. subtracting out this divergent term from the finite-temperature free energy function) to see if there is any remaining new physics to uncover. There are 2 common ways in which this can be done:

1. the first of which (being rather crude since Lorentz invariance is lost) is by asserting a cutoff value of $\boldsymbol{k}$ (*cutoff regularization*);
2. a more elegant solution is to exploit the fact that the dimension of the system D, has been taken as arbitrary through the derivations above. This will allow us to work in non-integer dimensions which as we will see, lead to a quelling of these divergences. This method is known as *dimensional regularization*.

> **Note**: Although more mathematically elegant, it is often difficult to extract physical insight using dimensional regularization as opposed to cutoff regularization.

Cutoff Regularization

To deal with the free scalar field divergence, we will first utilize the cutoff regularization technique. From Eq. (2.19), we consider a more physically intuitive observable $p(T)$, which represents the pressure. This, from thermodynamics, is known to follow the relation

$$\begin{aligned} & p(T) = \frac{T}{V} \ln Z \\ \Rightarrow \quad & p_{\text{free}}(T) = -\int \frac{d^D \boldsymbol{k}}{(2\pi)^D} \left[\frac{E_k}{2} + T \ln \left(1 - e^{-\beta E_k}\right) \right]. \end{aligned} \tag{2.21}$$

This quantity, is of course still divergent, even in the zero temperature limit which take the form

$$p(0) = -\frac{1}{2} \int \frac{d^D \boldsymbol{k}}{(2\pi)^D} E_k = -\frac{1}{2} \int \frac{d^D \boldsymbol{k}}{(2\pi)^D} \sqrt{\boldsymbol{k}^2 + m^2}. \tag{2.22}$$

Motivated by the pressure, we now consider instead a more generalized version of it in the zero temperature limit, by introducing a parameter A. We define this quantity as

$$\Phi(m, D, A) \equiv \int \frac{d^D\boldsymbol{k}}{(2\pi)^D} \left(\boldsymbol{k}^2 + m^2\right)^{-A} \tag{2.23}$$

$$\Rightarrow \quad p(0) = -\frac{1}{2}\Phi\left(m, D, -\frac{1}{2}\right). \tag{2.24}$$

Notice that the integral in $\Phi(m, D, A)$ can be evaluated rather simply, by noticing that the integrand is isotropic. As such, moving to hyperspherical coordinates grants

$$\Phi(m, D, A) = \frac{\Omega_D}{(2\pi)^D} \int_0^\infty \left(k^2 + m^2\right)^{-A} k^{D-1} dk, \tag{2.25}$$

where $\Omega_D = 2\pi(D/2)/\Gamma(D/2)$ is the solid angle in D dimensions (e.g. $\Omega_3 = 4\pi$). Notice that so long as m is nonzero, the integrand remains well-behaved, while the integral only divergence for large wave-number ($k \to \infty$). This (as analogous to the ultraviolet catastrophe) is known as *UV divergence* (the converse case of divergences for low wave-numbers are known as *IR divergences*). To deal with the UV divergence, we cutoff the k at some value $\Lambda \gg 1$, which grants

$$\int_0^\Lambda \left(k^2 + m^2\right)^{-A} k^{D-1} dk \sim \int k^{-2A} k^{D-1} dk \bigg|_{k=\Lambda} \sim \frac{\Lambda^{D-2A}}{D-2A}. \tag{2.26}$$

In the case where $\{D, A\} = \{3, -1/2\}$, this grants $p(0) \sim \Lambda^4$. As such, we say that this divergence "goes as degree 4". Without assuming that Λ is large, we can in fact solve for $p(0)$ analytically which gives

$$p(0) = -\frac{m^4}{16\pi^2}\left[\frac{\Lambda^4}{m^4} + \frac{\Lambda^2}{2m^2} - \frac{1}{2}\ln\left(\frac{2\Lambda}{m}\right)\right], \tag{2.27}$$

showing that there is in fact a sub-divergence of degree 2, and a sub-sub logarithmic divergence.

Dimensional Regularization

Let us instead now utilize dimensional regularization for this same problem and see what emerges (hint, renormalization will take a leading role here for things to work out). The integral for Φ for arbitrary D can in fact be solved analytically here, to give the form

$$\begin{aligned}\Phi(m, D, A) &= \frac{\Omega_D}{(2\pi)^D} \frac{\Gamma\left(A - \frac{D}{2}\right)\Gamma\left(1 + \frac{D}{2}\right)}{D\Gamma(A)} \left(m^2\right)^{-A+\frac{D}{2}} \\ &= \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(A - \frac{D}{2}\right)}{\Gamma(A)} \left(m^2\right)^{-A+\frac{D}{2}}.\end{aligned} \tag{2.28}$$

A problem arises however for $\{D, A\} = \{3, -1/2\}$, since this would render the argument of the gamma function a negative integer -2, and is thus divergent. One possible way to assess this is to instead consider the case where D is not an integer (almost but not quite 3 dimensions), but defined by

$$D = 3 - 2\varepsilon, \tag{2.29}$$

which we can then take the limit as $\varepsilon \to 0$ at the end of the computation. Using this, Φ becomes

$$\Phi\left(m, 3-2\varepsilon, -\frac{1}{2}\right) = \frac{1}{(4\pi)^{3/2-\varepsilon}} \frac{\Gamma(-2+\varepsilon)}{\Gamma(-1/2)} (m^2)^{2-\varepsilon}, \tag{2.30}$$

which will remain finite for any $\varepsilon \in (0,1)$. Using the Gamma function property $x\Gamma(x) = \Gamma(1+x)$ and Taylor expanding in ε, we end up with

$$p(0) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} - \ln\big(m^2\big) + \ln(4\pi) - \gamma_E + \frac{3}{2} + \mathcal{O}(\varepsilon)\right], \tag{2.31}$$

where $\gamma_E \approx 0.577$ is the Euler's constant. A distinct peculiarity arises in this derivation, as we see that there is a non-unit free argument in one of the logarithmic terms, $\ln\big(m^2\big)$. The appearance of this term can be traced back to the integral measure (differential element) being non-integer. To fix this, we can add a scaling parameter μ, (with units of mass) raised to the appropriate power that fixes this problem of dimensionality

$$\Phi(m, 3-2\varepsilon, A) \equiv \mu^{2\varepsilon} \int \frac{d^{3-2\varepsilon}\boldsymbol{k}}{(2\pi)^{3-2\varepsilon}} \left(\boldsymbol{k}^2 + m^2\right)^{-A}. \tag{2.32}$$

This added factor μ, is referred to as the *renormalization scale parameter*, which modifies the zero temperature pressure as

$$p(0) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} - \ln\left(\frac{\mu^2}{m^2}\right) + \ln(4\pi) - \gamma_E + \frac{3}{2} + \mathcal{O}(\varepsilon)\right]. \tag{2.33}$$

More conveniently, we can absorb several factors into the renormalization scale parameter using the logarithmic laws such that

$$\overline{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E} \tag{2.34}$$

$$\Rightarrow \quad p(0) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} - \ln\left(\frac{\overline{\mu}^2}{m^2}\right) + \frac{3}{2} + \mathcal{O}(\varepsilon)\right]. \tag{2.35}$$

The introduction of this scale parameter in this way is referred to as the "$\overline{\text{MS}}$ (minimal subtraction) scheme".

§2.3.3 Renormalization

We have now derived the zero temperature pressure under both regularization schemes (Eqs. 2.27, 2.35) and although both divergent, seem to take on very different forms. In fact the only thing that is identical between these 2 expressions (which we present again below for clarity), is the prefactor $m^4/(64\pi^2)$.

$$\text{cutoff regularization}: \quad p(0) = -\frac{m^4}{16\pi^2}\left[\frac{\Lambda^4}{m^4} + \frac{\Lambda^2}{2m^2} - \frac{1}{2}\ln\left(\frac{2\Lambda}{m}\right)\right], \tag{2.36}$$

$$\text{dimensional regularization}: \quad p(0) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} - \ln\left(\frac{\overline{\mu}^2}{m^2}\right) + \frac{3}{2} + \mathcal{O}(\varepsilon)\right]. \tag{2.37}$$

This is not looking good, since our value for the pressure cannot be regularization scheme dependent! Before we scrap our work and lose all hope, let us first go back to the finite temperature case. At finite temperatures, we can write the pressure in terms of the zero temperature pressure and a finite temperature correction

$$\begin{aligned} p(T) &= p(0) - J_B(T,m), \\ \text{where} \quad J_B(T,m) &\equiv T \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \ln\left(1 - e^{-\beta E_k}\right). \end{aligned} \tag{2.38}$$

The integral J_B is in fact non-divergent, and can even be calculated analytically in the zero mass limit as $J_B(T, m = 0) = -\pi^2 T^4/90$. So if one were to just look for the difference $p(T) - p(0)$, this would give a perfectly reasonable solution. However even in this case, there is ambiguity as to how $p(0)$ is defined since it a divergent, leaving room for $p(T) - p(0)$ to be well defined up to a constant. So we ask, is there a more systematic fashion to deal with these divergences?

Well the answer would have to be yes (otherwise there really wouldn't be much else to talk about), for which we call this procedure *renormalization*. For this, we turn our eyes all the way back to the Lagrangian. Recall that a Lagrangian preserves the physics up to a constant (which is of course Lorentz invariant), so we can write our Lagrangian as

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi) - K, \tag{2.39}$$

where K is a constant known as a *counterterm*. The action for the free scalar field Lagrangian including this added factor of K would then become

$$S_E = \int d^4x_E \left[\frac{1}{2}\partial_a\phi\partial_a\phi + \frac{m^2}{2}\phi^2 + K\right], \tag{2.40}$$

$$\begin{aligned} \Rightarrow \quad p^{\text{renorm}}(T) &= -K - \int \frac{d^D\boldsymbol{k}}{(2\pi)^D}\left[\frac{E_k}{2} + T\ln\left(1 - e^{-\beta E_k}\right)\right] \\ &= -K + p(0) - J_B(T,m). \end{aligned} \tag{2.41}$$

It is now apparent, that we can simply choose K to negate divergent terms in $p(0)$, which would leave

$$\boxed{p^{\text{renorm}}(T) = -J_B(T,m) + \frac{m^4}{64\pi^2}\ln\left(\frac{\overline{\mu}^2 e^{3/2}}{m^2}\right)}, \tag{2.42}$$

a finite-valued quantity up to renormalization scale parameter $\overline{\mu}$. The result above was obtained by negating terms from the dimensionally regularized function, but can also be done to the cutoff regularized function to give an equivalent result.

Chapter 3

Interacting QFT

Thus far, we have worked with a quantum field theory in the absence of interactions (free). In this section, we aim to develop an interacting quantum field theory by means of perturbation theory. Interacting field theories can be constructed by generalizing the free scalar field theory above, but to potentials with higher powers of ϕ.

§3.1 Introduction

To start our discussion of interaction QFTs, we want a way to categorize the terms in the Lagrangian in various powers of ϕ. To do so, we first classify the different orders of ϕ by their mass dimension (which is the only dimension left after setting $c = \hbar = k_B = 1$). Recall that the Euclidean action is given as

$$S_E = \int d^4 x_E \left[\frac{1}{2} \partial_a \phi \partial_a \phi + V(\phi) \right]. \tag{3.1}$$

The action is a dimensionless quantity with the integral measure having a mass dimension of -4, which implies that all terms in the integrand must have mass dimension 4, i.e.

$$[\partial_a \phi \partial_a \phi]_m = [V(\phi)]_m = 4 \tag{3.2}$$

$$\Rightarrow \quad [\phi]_m = 1. \tag{3.3}$$

> **Note**: Although ϕ is a classical field, once it is used in the path integral, it is promoted into a quantum operator. The mass dimension of a quantum operator may differ from its classical field, for which we call the classical field mass dimension “naive”.

There is a standard nomenclature associated to the mass dimension of terms (operators) in the potential. Firstly, the coefficients to these oprators (polynomic field terms) are known as *coupling constants*. If the mass dimension of the operator is less than the number of spactime dimensions n (e.g. $\phi^0, \ldots, \phi^3$ for $n = 4$), then this operator is known as *relevant*. If however the mass dimension is equal to n, these operators are known as *marginal*. Lastly, if operators have mass dimension exceeding n, these are known as *irrelevant operators*.

A good starting point for constructing an interacting QFT is by adding a marginal operator into the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4, \tag{3.4}$$

where $\lambda \geq 0$ is a unit-free coupling constant. This potential will be the "poster child" of sorts, for interacting QFTs. We then write the Euclidean action as consisting of 2 terms

$$S_0 = \int d^4x_E \left[\frac{1}{2}\partial_a\phi\partial_a\phi + \frac{1}{2}m^2\phi^2\right], \tag{3.5a}$$

$$S_I = \lambda \int d^4x_E\phi^4. \tag{3.5b}$$

This renders the partition function as

$$Z = \int \mathcal{D}_\phi e^{-S_0 - S_I}. \tag{3.6}$$

Recall that to solve these path integrals, we first consider the discretized version with path integral measure

$$\mathcal{D}_\phi \quad \to \quad \prod_{j=1}^{N} d\phi_j, \tag{3.7}$$

then take the limit as $N \to \infty$ after evaluation of the integrals. Unfortunately, the lack of quadratic dependence in S_I disallows the method of Fourier transforms to nicely evaluate these integrals as we did before. So, we turn to perturbation theory to solve these integrals approximately. Expanding the exponent into powers of λ, gives a *perturbative series*

$$\begin{aligned} Z &= \int \mathcal{D}_\phi e^{-S_0}\left(1 - S_I + \frac{1}{2}S_I^2 - \frac{1}{3!}S_I^3 + \dots\right) \\ &= Z_{\text{free}}\left(1 - \langle S_I\rangle + \frac{1}{2}\langle S_I^2\rangle - \frac{1}{3!}\langle S_I^3\rangle + \dots\right), \end{aligned} \tag{3.8}$$

where

$$Z_{\text{free}} = \int \mathcal{D}_\phi e^{-S_0}, \tag{3.9}$$

$$\langle \dots \rangle \equiv \frac{1}{Z_{\text{free}}}\int \mathcal{D}_\phi \left(\dots\right) e^{-S_0}. \tag{3.10}$$

The second identity above is the quantum mechanical *path integral expectation value* taken over an observable. It is not guaranteed that this perturbative series converges, but it is often the case that it will at least be asymptotic which will allow an approximation by truncation of the series so long as λ as small.

§3.1.1 First-Order QFT and Wick's Theorem

Let's consider the perturbative series just to first-order. In this case, we have

$$\begin{aligned} Z &\approx Z_1 \\ &= Z_{\text{free}} \left(1 - \lambda \int d^4 x_E \langle \phi^4(x) \rangle \right). \end{aligned} \tag{3.11}$$

We see now that we will have to compute the quantity $\langle \phi^4(x) \rangle$, and eventually $\langle \phi^4(x_1) \dots \phi^4(x_m) \rangle$ for higher order terms where m is the order of the term. To evaluate this, we are going utilize *Wick's theorem.*

Theorem 3.1.1. Wick's theorem: *The expectation value of a product of operators taken over a Gaussian action can be decomposed as a sum over all combinations of operator pair expectation values (two-point functions)*

$$\langle \phi(x_1)\phi(x_2)\dots\phi(x_n) \rangle = \sum_{\text{combinations}} \langle \phi(x_1)\phi(x_2) \rangle \dots \langle \phi(x_{n-1})\phi(x_n) \rangle. \tag{3.12}$$

A simple example of this theorem would be the ϕ to the fourth operator, which then can be written as quadratic two-point functions

$$\begin{aligned} \langle \phi^4(x) \rangle &= \sum \langle \phi^2(x) \rangle \langle \phi^2(x) \rangle \\ &= 3 \langle \phi^2(x) \rangle^2. \end{aligned} \tag{3.13}$$

We now present the proof of this theorem.

Proof. Consider a Gaussian integral over a vector field v_i

$$\int \exp\left(-\frac{1}{2} v_i A_{ij} v_j + b_i v_i\right) d\boldsymbol{v}, \tag{3.14}$$

where A_{ij} is a positive-definite matrix so that the integral converges, and b_i is an arbitrary vector. We then define some function $W[b]$, such that

$$e^{W[b]} \equiv \int \exp\left(-\frac{1}{2} v_i A_{ij} v_j + b_i v_i\right) d\boldsymbol{v}, \tag{3.15}$$

and call $e^{W[b]}$ the *generating function.* We can then evaluate the integral through variable substitution

$$u_i \equiv v_i + A^{-1}_{ij} b_j \tag{3.16}$$

$$\begin{aligned} \Rightarrow \quad e^{W[b]} &= e^{-\frac{1}{2} b_i A^{-1}_{ij} b_j} \int \exp\left(-\frac{1}{2} u_i A_{ij} u_j\right) d\boldsymbol{u} \\ &= e^{-\frac{1}{2} b_i A^{-1}_{ij} b_j} e^{W[0]}. \end{aligned} \tag{3.17}$$

We can now use the Feynman trick and take the derivative of the generating function with respect to b_i, which will give

$$\begin{aligned}\frac{\partial}{\partial b_i}e^{W[b]} &= \int d\boldsymbol{v}\frac{\partial}{\partial b_i}\exp\left(-\frac{1}{2}v_iA_{ij}v_j + b_iv_i\right)\\ &= \int \exp\left(-\frac{1}{2}v_iA_{ij}v_j + b_iv_i\right)v_i d\boldsymbol{v},\end{aligned} \tag{3.18}$$

$$\Rightarrow \quad \frac{\partial}{\partial b_i}e^{W[b]}\bigg|_{b=0} = \langle v_i\rangle e^{W[0]} \tag{3.19}$$

$$\Rightarrow \quad \frac{\partial}{\partial b_i\partial b_j\dots\partial b_n}e^{W[b]}\bigg|_{b=0} = \langle v_iv_j\dots v_n\rangle e^{W[0]} \tag{3.20}$$

$$\Rightarrow \quad \langle v_iv_j\dots v_n\rangle = \frac{\partial}{\partial b_i\partial b_j\dots\partial b_n}e^{-\frac{1}{2}b_iA^{-1}_{ij}b_j}\bigg|_{b=0}. \tag{3.21}$$

From this, we see several useful properties of these expecation values:

1. $\langle 1\rangle = 1$;
2. the expectation of an odd number of operators is zero

$$\langle v_i\rangle = \langle v_iv_jv_k\rangle = \dots = 0;$$

From the second property, we will only need to consider the even numbered operator products, which we compute as

$$\langle v_iv_j\rangle = A^{-1}_{ij} \tag{3.22}$$

$$\begin{aligned}\langle v_iv_jv_mv_n\rangle &= A^{-1}_{ij}A^{-1}_{mn} + A^{-1}_{im}A^{-1}_{jn} + A^{-1}_{in}A^{-1}_{jm}\\ &= \langle v_iv_j\rangle\langle v_mv_n\rangle + \langle v_iv_m\rangle\langle v_jv_n\rangle + \langle v_iv_n\rangle\langle v_jv_m\rangle\end{aligned} \tag{3.23}$$

$$\vdots$$

$$\langle v_iv_j\dots v_n\rangle = \sum_{\text{combinations}}\langle v_1v_2\rangle\dots\langle v_{n-1}v_n\rangle. \tag{3.24}$$

Now taking the limit as the vectors v_i have dimensions that tend to infinity, this becomes a continuous scalar field $v_i \to \phi(x)$, which produces the result in Wick's theorem

$$\langle\phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = \sum_{\text{combinations}}\langle\phi(x_1)\phi(x_2)\rangle\dots\langle\phi(x_{n-1})\phi(x_n)\rangle. \tag{3.25}$$

□

§3.2 Feynman Diagrams (A First Look)

Going back to the perturbative series for interacting quantum field theories, we see that the terms quickly get messy, making it hard to keep track of terms while considering the physics behind such processes. In this section, we will develop a graphical method of book keeping such terms in the perturbative series known as *Feynman diagrams*. The goal of these diagrams are to encode all the relevant information that would otherwise have to be written out as tedious integrals. To begin with, we consider the pressure, which can be derived from the partition function as

$$
\begin{aligned}
p &= \frac{1}{\beta V}\ln Z \\
&= \frac{1}{\beta V}\ln\left[Z_{\text{free}}\left(1-\langle S_I\rangle+\frac{1}{2}\langle S_I^2\rangle-\frac{1}{3!}\langle S_I^3\rangle+\dots\right)\right] \\
&\approx \frac{1}{\beta V}\ln Z_{\text{free}}+\frac{1}{\beta V}\left[-\langle S_1\rangle+\frac{1}{2}\left(\left\langle S_1^2\right\rangle-\langle S_1\rangle^2\right)-\frac{1}{3!}\left(\left\langle S_1^3\right\rangle-3\left\langle S_1^2\right\rangle\langle S_1\rangle+2\langle S_1\rangle^3\right)\right] \\
&\equiv p_{\text{free}}+p_{(1)}+p_{(2)}+p_{(3)}+\dots,
\end{aligned}
\tag{3.26}
$$

where to be explicit, we defined

$$p_{\text{free}} \equiv \frac{1}{\beta V}\ln Z_{\text{free}} \tag{3.27a}$$

$$p_{(1)} \equiv -\frac{1}{\beta V}\langle S_1\rangle \tag{3.27b}$$

$$p_{(2)} \equiv \frac{1}{2\beta V}\left(\left\langle S_1^2\right\rangle-\langle S_1\rangle^2\right) \tag{3.27c}$$

$$p_{(3)} \equiv -\frac{1}{3!\beta V}\left(\left\langle S_1^3\right\rangle-3\left\langle S_1^2\right\rangle\langle S_1\rangle+2\langle S_1\rangle^3\right) \tag{3.27d}$$

$$\vdots$$

Considering just the first-order perturbative term, we have from Wick's theorem that

$$p_{(1)} = -\frac{3\lambda}{\beta V}\int d^4x\,\langle\phi(x)\phi(x)\rangle^2\,. \tag{3.28}$$

The integrand in general ($\langle\phi(x)\phi(y)\rangle$), is a measure of how information is passed through the field from point x to point y. This is often referred to as the *(free) propagator* (a.k.a. *(free two-point function)*) and graphically denoted simply as a line (from x to y)

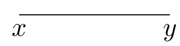
$\langle\phi(x)\phi(y)\rangle =$ 

Figure 3.1: Feynman diagram for the free propagator $\langle\phi(x)\phi(y)\rangle$.

For the first-order pressure term, we see that the free propagator takes information from x back to the same position, hence it will be a closed-loop. It also has 2 of such propagators (since it is squared), so it is denoted as

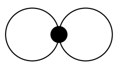
$\langle\phi(x)\phi(x)\rangle^2 \sim$ 

Figure 3.2: Feynman diagram for the free propagator $\langle\phi(x)\phi(x)\rangle^2$ with coupling λ.

where the solid dot denotes the coupling λ. The solid dot will always denote a coupling and so must always be at a vertex:

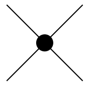
$-\lambda =$ 

Figure 3.3: Feynman diagram a coupling λ.

§3.2.1 Combinatorial Factor

For the coupling vertex illustrated in Fig. 3.3, there are in fact several ways in which such vertices could come about. As such, these permutations must be considered and gives to a *combinatorial factor*. To better understand this, we consider the following example.

Example:

Starting with the coupling vertex, we see that if we were to take one of the "legs" and connect it to another leg, there are 3 possible ways to do so. So in fact, we have that

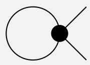
 $= -3\lambda\langle\phi(x)\phi(x)\rangle$

Figure 3.4: Single coupling vertex with 2 connected legs.

However, if we not pick another leg and attempt to connect it, there is only one way in which we can do so. As such, connecting the remaining legs does not add any additional combinatorial factors, but does add another propagator such that

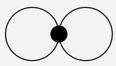
 $= -3\lambda\langle\phi(x)\phi(x)\rangle\langle\phi(x)\phi(x)\rangle$

Figure 3.5: Single coupling vertex with 4 connected legs.

So we see that with the vertex and propagator rules, the 2 loop single vertex diagram above does indeed retrieve the first-order pressure correction term (up to relevant factors of βV).

Note: The *Feynman rules* for drawing Feynman diagrams can be formulated in both position and momentum space. They are very helpful in dealing with perturbative series ($\lambda \ll 1$), but are difficult (although not impossible) to construct for non-perturbative results.

To better appreciate the utility of Feynman diagrams, let us try construct the Feynman diagram for the second-order term in the pressure perturbative series. We have that

$$p_{(2)} \equiv \frac{1}{2\beta V}\left(\langle S_l^2\rangle - \langle S_1\rangle^2\right). \tag{3.29}$$

For these terms, there are going to be 2 vertices and hence, we can draw all the Feynman diagrams, first for $\langle S_I^2\rangle$ as

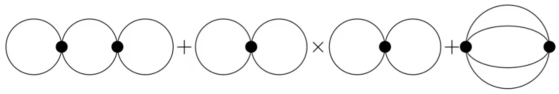


Figure 3.6: Feynman diagram for $\langle S_I^2 \rangle$.

The second diagram above (with the multiplication sign) denotes a *disconnected diagram*, in which no propagator connects the 2 vertices. The converse of this is then a *connected diagram.* A result of this is that the corresponding integral factorizes. The term disconnected diagram is used for all diagrams with at least one vertex that is not connected to the rest of the diagram. We can then see that the Feynman diagram for $\langle S_I \rangle^2$ would just be the disconnected diagram in Fig. 3.6. The resulting Feynman diagram for $p_{(2)}$ would thus be:

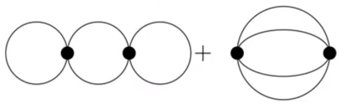


Figure 3.7: Feynman diagram for $\langle S_I^2 \rangle$.

since the disconnected diagrams cancel out.

Note: This cancelling out is in fact **not** a coincidence, as it can be shown that **all** physical observables do **not** have disconnected diagrams. This leads to major simplifications in higher-order perturbative series calculations.

§3.3 The Free-Scalar Field Propagator

Recall that the free propagator was constructed as

$$\langle \phi(x)\phi(y) \rangle = \frac{\int \mathcal{D}_\phi e^{-S_0} \phi(x)\phi(y)}{Z_{\text{free}}} \tag{3.30}$$

$$\text{where} \quad S_0 = \frac{1}{2} \int d^4 x_E \left[\partial_a \phi \partial_a \phi + m^2 \phi^2 \right], \tag{3.31}$$

$$Z_{\text{free}} = \int \mathcal{D}_\phi e^{-S_0}. \tag{3.32}$$

Since the action is quadratic in the field, the integral is Gaussian, which result involves the inverse of the operator $\partial_a \partial_a + m^2$. This turns out to be more easily expressed in momentum space, for which we once again employ the Fourier transform trick involving Matsubara frequencies, giving us

$$\phi(x) = \frac{T}{V} \sum_{\omega_n, \boldsymbol{k}} e^{i\omega_n \tau + i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{\phi}(\omega_n, \boldsymbol{k}), \tag{3.33}$$

where the Fourier components are denoted as $\tilde{\phi}$. For lighter notation, we will define $\boldsymbol{K} = (\omega_n, \boldsymbol{k})$

as the Euclidean 4-momentum. Plugging this back into the free propagator gives

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{\beta^2V^2}\sum_{\boldsymbol{P},\boldsymbol{K}} \frac{\int \mathcal{D}_{\tilde{\phi}} e^{-S_0}\tilde{\phi}(\boldsymbol{P})\tilde{\phi}(\boldsymbol{K})e^{i\boldsymbol{P}\cdot\boldsymbol{x}+i\boldsymbol{K}\cdot\boldsymbol{y}}}{Z_{\text{free}}}. \tag{3.34}$$

Note that additional Jacobian factors drop out when divided by the free partition function. Recall that the action in Fourier space is written as

$$S_0 = \frac{1}{2\beta V}\sum_{\boldsymbol{K}}(K^2+m^2)\left|\tilde{\phi}(\boldsymbol{K})\right|^2. \tag{3.35}$$

From the eveness of a Gaussian function, we have that $\int d\boldsymbol{v} e^{-\frac{1}{2}v_iA_{ij}v_j}v_mv_n$ is only non-vanishing when $m=n$, which implies that the propagator is non-trivial only when $\tilde{\phi}(\boldsymbol{p})\tilde{\phi}(\boldsymbol{K}) = \left|\tilde{\phi}(\boldsymbol{K})\right|^2$. Periodic boundary conditions also dictate that $\tilde{\phi}(\boldsymbol{P}) = \tilde{\phi}^*(-\boldsymbol{P})$, which when combined with the previous condition, implies $\boldsymbol{P}+\boldsymbol{K}=0$ for non-trivial propagators. As such, we have that

$$\int \mathcal{D}_{\tilde{\phi}} \exp\left(-\frac{1}{2\beta V}\sum_{\boldsymbol{Q}}(Q^2+m^2)\left|\tilde{\phi}(\boldsymbol{Q})\right|^2\right)\tilde{\phi}(\boldsymbol{P})\tilde{\phi}(\boldsymbol{K}) = Z_{\text{free}}\times\left(\frac{\beta V}{K^2+m^2}\right)\delta_{\boldsymbol{P},-\boldsymbol{K}}, \tag{3.36}$$

where $\delta_{i,j}$ is the Kronecker delta function. This therefore simplifies the free propagator to

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{\beta V}\sum_{\boldsymbol{K}}\frac{e^{i\boldsymbol{K}\cdot(\boldsymbol{y}-\boldsymbol{x})}}{K^2+m^2} = \frac{1}{\beta V}\sum_{\boldsymbol{K}}\frac{e^{i\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{K^2+m^2}, \tag{3.37}$$

where the equality follows from $\sum_{\boldsymbol{K}} = \sum_{-\boldsymbol{K}}$.

> **Note**: The free propagator only depends on the difference in position $\boldsymbol{x}-\boldsymbol{y}$, which is expected to emerge from the translational invariance of the operator $\Box + m^2$.

In the large volume limit ($V\to\infty$), the sum over wavevectors $\boldsymbol{k}$, becomes an integral

$$\langle\phi(x)\phi(y)\rangle = T\sum_{\omega_n}\int\frac{d^Dk}{(2\pi)^D}\frac{e^{i\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{K^2+m^2}. \tag{3.38}$$

Propagators are also sometimes referred to as *Green's functions*, denoted as $G(\boldsymbol{x},\boldsymbol{y})$. In the case of the free propagator, it is the Green's function of the *Klein-Gordon equation*, for which it is written as

$$\boxed{G_{\text{free}}(\boldsymbol{x}-\boldsymbol{y}) = \langle\phi(x)\phi(y)\rangle = T\sum_{\omega_n}\int\frac{d^Dk}{(2\pi)^D}\frac{e^{i\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{K^2+m^2}}. \tag{3.39}$$

In the zero-temperature limit, the sum over Matsubara frequencies also tends to an integral, granting us

$$\lim_{T\to 0} G_{\text{free}}(\boldsymbol{x}-\boldsymbol{y}) = \int\frac{d^{D+1}k}{(2\pi)^{D+1}}\frac{e^{i\boldsymbol{K}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{K^2+m^2}. \tag{3.40}$$

With this, we now have the means to compute the perturbative pressure for an interacting QFT. For $\langle\phi(x)\phi(x)\rangle = G_{\text{free}}(\mathbf{0})$, we utilize a trick to evaluate this integral which relies on the identity

$$\begin{aligned}\frac{1}{K^2+m^2} &= \frac{\partial}{\partial(m^2)}\ln\left(K^2+m^2\right)\\ \Rightarrow\quad G_{\text{free}}(\mathbf{0}) &= \frac{\partial}{\partial(m^2)}T\sum_{\omega_n}\int\frac{d^Dk}{(2\pi)^D}\ln\left(\omega_n^2+k^2+m^2\right).\end{aligned} \tag{3.41}$$

Recalling from Eq. (1.44) that the free partition function can be written as

$$\begin{aligned}Z_{\text{free}} &= \prod_{\boldsymbol{k}}\frac{T}{E_k}\prod_{n=1}^{\infty}\frac{\omega_n^2}{\omega_n^2+E_k^2}\\ \Rightarrow\quad \ln Z_{\text{free}} &= -\frac{1}{2}\sum_{\boldsymbol{k}}\sum_{n}\ln\left(\omega_n^2+k^2+m^2\right)+m+\dots,\end{aligned} \tag{3.42}$$

where the other terms (denoted "...") are independent of m and not relevant to this calculation. Comparing this with the Green's function, we get that

$$G_{\text{free}}(\mathbf{0}) = -2\frac{\partial p_{\text{free}}}{\partial(m^2)}, \tag{3.43}$$

where we have already evaluated the free pressure via dimensional regularization in Eq. (2.35). Writing this out explicitly, we have

$$\boxed{G_{\text{free}}\left(0\right) = -\frac{m^2}{16\pi^2}\left[\frac{1}{\varepsilon}+\ln\left(\frac{\bar{\mu}^2e^{\frac{1}{2}}}{m^2}\right)\right]+I_B(T,m)}, \tag{3.44}$$

$$\text{where}\quad I_B(T,m)\equiv 2\frac{\partial}{\partial(m^2)}J_B(T,m) = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{E_k(e^{\beta E_k}-1)}. \tag{3.45}$$

It is clear that this result is again divergent, saying that the free propagator diverges when $x\to 0$ (which is a UV-divergence since small distances correspond to high frequencies). However, this divergence is surpressed when we set $m=0$ at $D=3$, which gives

$$\begin{aligned}\lim_{m\to 0}G_{\text{free}}\left(0\right) &= I_B(T,0)\\ &= \int\frac{d^3k}{(2\pi)^3}\frac{1}{k(e^{\beta k}-1)}\\ &= \frac{T^2\zeta(2)}{2\pi^2}.\end{aligned} \tag{3.46}$$

As such, we have the first order correction to the pressure in $(3+1)$-dimensions with a ϕ^4 interaction being

$$\boxed{p_{(1)} = p_{\text{free}} - 3\lambda G_{\text{free}}^2(0)}. \tag{3.47}$$

For a massless scalar field ($m=0$), this result takes the succinct form

$$p_{(1)} = \frac{\pi^2T^4}{90} - \lambda\frac{T^4}{18}. \tag{3.48}$$

§3.3.1 Zero Temperature Free Propagator

In this section, we will be taking a closer look at the free propagator in the zero-temperature limit to understand and interpret it. As presented earlier, this is written as

$$\lim_{T\to 0} G_{\text{free}}(\boldsymbol{x}) = \int \frac{d^{D+1}k}{(2\pi)^{D+1}} \frac{e^{i\boldsymbol{K}\cdot x}}{K^2+m^2}. \tag{3.49}$$

We can see that the integral is in the form of a Fourier transform, so the $T = 0$ free propagator in momentum space is simply written as

$$\lim_{T\to 0} \tilde{G}_{\text{free}}(\boldsymbol{x}) = \frac{1}{K^2+m^2}, \tag{3.50}$$

where again, $K^2 = \omega_n^2 + k^2$ with $\omega_n = 2\pi nT$ being the Matsubara frequencies ($n = 0, \pm 1, \pm 2, \ldots$). We can write K^2 in a Lorentz invariant form by appending an i to ω_n, which reads

$$K^2 = -(i\omega_n)^2 + k^2. \tag{3.51}$$

In Minkowski space, the 4-momentum is given as $k^\mu = (k^0, \boldsymbol{k})$, so we can attempt to utilize *analytic continuation* and extend

$$i\omega_n \quad \to \quad k^0 + i\varepsilon^+, \tag{3.52}$$

where ε^+ is a small positive value which is later taken to 0. For the $T = 0$ free propagator in momentum space, analytic continuation as done above leads to the retarded Green's function in Minkowski space

$$\begin{aligned}\tilde{G}_{\text{R, free}}(k^0, \boldsymbol{k}) &= \tilde{G}_{\text{R, free}}(-ik^0 + \varepsilon^+, \boldsymbol{k}) \\ &= \frac{1}{-(k^0+i\varepsilon^+)^2 + k^2 + m^2} \\ &= \frac{1}{k_\mu k^\mu + m^2 - i\varepsilon^+ \text{sign}(k^0)}.\end{aligned} \tag{3.53}$$

Further separating out the real and imaginary parts gives us

$$\tilde{G}_{\text{R, free}}(k^0, \boldsymbol{k}) = \mathcal{P}\left[\frac{1}{k_\mu k^\mu + m^2}\right] + i\left[\frac{\varepsilon^+ \text{sign}(k^0)}{(k_\mu k^\mu + m^2)^2 + (\varepsilon^+)^2},\right] \tag{3.54}$$

where $\mathcal{P}$ indicates the Cauchy principle value. We can now take the limit in which the small positive regulator is taken to zero, and use the identity

$$\lim_{\varepsilon^+\to 0} \frac{\varepsilon^+}{x^2+\varepsilon^+} = \pi\delta(x) \tag{3.55}$$

$$\Rightarrow \quad \tilde{G}_{\text{R, free}}(k^0, \boldsymbol{k}) = \mathcal{P}\left[\frac{1}{k_\mu k^\mu + m^2}\right] + i\frac{\pi}{2E_k}\left[\delta(k^0 - E_k) - \delta(k^0 + E_k)\right], \tag{3.56}$$

where $E_k = k^2 + m^2$. The real and imaginary parts of this function are not unrelated, but for now we shall just focus on the imaginary part which is referred to as the *spectral function*:

$$\boxed{\tilde{\rho}_{\text{free}}(k^0, \boldsymbol{k}) = \text{Im}\, \tilde{G}_{\text{R, free}}(k^0, \boldsymbol{k}) = \frac{\pi}{2E_k}\left[\delta(k^0 - E_k) - \delta(k^0 + E_k)\right]}, \tag{3.57}$$

which we see is sharply peaked at energies $\pm E_k$. This sort of peaked behavior not only appears in QFT, but also in classical systems for which the energy behavior is localized at (or around) some E_k, forming what is known as a *quasiparticle*. The localization at $-E_k$ on the other hand can for now be taken at face value and labeled an *anti-quasiparticle*. Anti-particles as we will see, natural emerge in QFT.

The structure of this spectral function suggest that the relevant excitations in a free QFT are particle-like, which the quasiparticles and anti-quasiparticles satisfy the dispersion relation

$$(k^0)^2 = k^2 + m^2, \tag{3.58}$$

that implies that quasiparticles have mass, m. In other words, the poles of the analytically continued free propagator corresponds to the quasiparticle mass. Dispersion relations can in general be measured and used to infer the properties of quasiparticles, especially in more complex interacting systems. Dispersion relations can also have imaginary parts, which pertain to the lifetime of the quasiparticle. In this case (free QFT), the quasiparticle would thus have an infinite lifetime (unconditionally stable).

§3.4 Application: Thermal Phase Transitions

In this section, we shall be looking at an application of the quantum field theory formalism we have developed thus far. In particular, will will be looking at the problem of *thermal phase transitions* which occur for instance, in QCD studies of the early universe and the Higgs mechanism. Unfortunately, a rigorous treatment of what we are about to discuss requires additional tools (e.g. gauge fields, fermions, etc) that we have yet to discuss, but we do have the necessary tools for a qualitative understanding. To start, we consider again a ϕ^4 potential but now with the sign on the ϕ^2 term flipped:

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \lambda\phi^4. \tag{3.59}$$

For $\lambda > 0$, we will have that the ϕ^4 term will still allow for convergent integrals and *stabilize* the QFT despite the imaginary mass.

> **Note**: Any Lagrangian with an imaginary mass terms signals an instability in the system.

This potential plotted against ϕ will then look (as a function of ϕ) like a cross section of the famed “Mexican hat”. In this case, since the $\lambda = 0$ scenario causes the system to be completely unstable, perturbation will no longer work since we would be simply expanding around the wrong groundstate. So instead, we need to expand around the stable minimum of the system. To do so, we can decompose the quantum field into a constant plus fluctuations

$$\phi(x) = \overline{\phi} + \phi'(x), \tag{3.60}$$

where we will refer to the constant $\overline{\phi}$, as the *mean field* term. This renders the path integral as

$$Z = \int d\overline{\phi} \int \mathcal{D}_{\phi'} e^{-\overline{S}[\overline{\phi}] - S'[\overline{\phi}, \phi']}, \tag{3.61}$$

where $S[\overline{\phi}+\phi'] = \overline{S}[\overline{\phi}] + S'[\overline{\phi},\phi']$. The convenience that the mean field term brings, is that it is no longer a path integral but now simply a regular integral over $\overline{\phi}$. If we perform the path integral, we are left with an effective potential term $U_{\text{eff}}(\overline{\phi})$, such that

$$Z = \int d\overline{\phi} e^{-\beta V U_{\text{eff}}(\overline{\phi})}, \tag{3.62}$$

where V is the volume of the Euclidean thermal cylinder. In the large volume (or low temperature) limit, the partition function to lowest order is just the minimum of the effective potential

$$Z \approx e^{-\beta V U_{\text{eff}}(\overline{\phi}_{\text{min}}) + \mathcal{O}(\ln(\beta V))}, \tag{3.63}$$

such that $dU_{\text{eff}}(\overline{\phi})/d\overline{\phi}|_{\overline{\phi}_{\text{min}}} = 0$. If we ignore all fluctuations (which is a drastic but useful approximation for extracting the physics), we have

$$Z = \int d\overline{\phi} e^{-\overline{S}[\overline{\phi}]}, \tag{3.64a}$$

$$\text{where} \quad \overline{S}[\overline{\phi}] = \int d\tau \int d^3x V(\overline{\phi}) = \beta V \times V(\overline{\phi}), \tag{3.64b}$$

$V(\overline{\phi})$ being just the classical potential. Minimizing the classical potential with respect to ϕ will give us that

$$\overline{\phi}_{\text{min}} = \pm \frac{m}{2\sqrt{\lambda}} \tag{3.65}$$

$$\Rightarrow \quad Z_{\text{mf}} = \beta V \left(\frac{m^4}{16\lambda}\right). \tag{3.66}$$

Note: The expectation value of the field in the mean field approximation is simply the mean field itself, i.e.

$$\langle \phi(x) \rangle_{\text{mf}} = \overline{\phi}. \tag{3.67}$$

Recall that in the high temperature limit we will have $T \gg m$, for which we can ignore the mass scale which grants that $\overline{\phi} \approx 0$. So we conclude that between the zero temperature and high temperature cases (where m goes from non zero to effectively zero), there will indeed be a (thermal) phase transition in which the expectation of the field (computed with the full action) is an *order parameter*.

§3.4.1 Beyond Mean Field

To do better than the lowest order approximation, we will have to go back and systematically add the fluctuation terms $\phi'(x)$, to the scalar field. By construction of the fluctuation term, we have that its integral over x will evaluate to zero (i.e. $\int_x \phi'(x) = 0$). Plugging in $\phi(x) = \overline{\phi} + \phi'(x)$ into the potential grants us

$$V(\overline{\phi} + \phi'(x)) = V(\overline{\phi}) + \left(-m^2\overline{\phi} + 4\lambda\overline{\phi}^3\right)\phi' + \frac{1}{2}\left(-m^2 + 12\lambda\overline{\phi}^2\right)\phi'^2 + \left(4\lambda\overline{\phi}\right)\phi'^3 + (\lambda)\phi'^4. \tag{3.68}$$

When used to compute the action, we see that the term linear in ϕ' would vanish, leaving us with

$$Z = \int d\overline{\phi} e^{-\beta V V(\overline{\phi})} \int \mathcal{D}_{\phi'} e^{-S'_0[\overline{\phi}] - S'_I[\overline{\phi}]}; \tag{3.69a}$$

$$\text{where} \quad S'_0[\overline{\phi}] \equiv \frac{1}{2} \int_x \left[\partial_a \overline{\phi} \partial_a \overline{\phi} + \left(-m^2 + 12\lambda \overline{\phi}^2 \right) \phi'^2 \right], \tag{3.69b}$$

$$S'_I[\overline{\phi}] = \int_x \left[\left(4\lambda \overline{\phi} \right) \phi'^3 + (\lambda) \phi'^4 \right]. \tag{3.69c}$$

We notice that S'_0 looks like the action for a non-interacting QFT with an effective mass

$$m^2_{\text{eff}}(\overline{\phi}) = -m^2 + 12\lambda \overline{\phi}^2. \tag{3.70}$$

So as per what we have done before, we can just use perturbation theory to compute the effect of adding the S'_I contribution into the path integral (it turns out that this will not be necessary to extract the necessary physics). There are 2 scenarios we now consider:

1. symmetric phase ($\overline{\phi} = 0$): in this case, the calculations for the fluctuations revert to standard perturbation theory;
2. symmetry broken phase ($\overline{\phi} \neq 0$): in this case, the calculations are modified from the presence of a non-zero field expectation value;

Now we get to actually computing the terms in the perturbative series. The leading order term would of course just yield the result from a free field theory but replacing m with m_{eff}:

$$\int \mathcal{D}_{\phi'} e^{-S'_0[\overline{\phi}]} = \exp\left(-\frac{\beta V}{2} \int \frac{d^D \boldsymbol{k}}{(2\pi)^D} \left[\sqrt{k^2 + m^2_{\text{eff}}} + 2T \ln\left(1 - e^{-\beta E_k}\right) \right] \right) = e^{\beta V p_{\text{free}}(T)}. \tag{3.71}$$

The renormalized free pressure from the $\overline{\text{MS}}$ scheme has already been computed which grants us that

$$U^{\text{ren}}_{\text{eff},0}(\overline{\phi}) = V(\overline{\phi}) + J_B(T, m_{\text{eff}}(\overline{\phi})) - \frac{m^4_{\text{eff}}(\overline{\phi})}{64\pi^2} \ln\left(\frac{\overline{\mu}^2 e^{3/2}}{|m^2_{\text{eff}}(\overline{\phi})|} \right). \tag{3.72}$$

To simplify the discussion, we consider the zero-temperature case which causes J_B to vanish and choose $\bar{\mu} = m$, leaving us with

$$U^{\text{ren}}_{\text{eff},0}(\overline{\phi}) = V(\overline{\phi}) - \frac{m^4_{\text{eff}}(\overline{\phi})}{64\pi^2} \ln\left(\frac{m^2 e^{3/2}}{|m^2_{\text{eff}}(\overline{\phi})|} \right). \tag{3.73}$$

We then compare this corrected potential to the zeroth-order mean field approximation by setting $\lambda = 1$ (for illustrative purposes), which gives us the plots in Fig. 3.8.

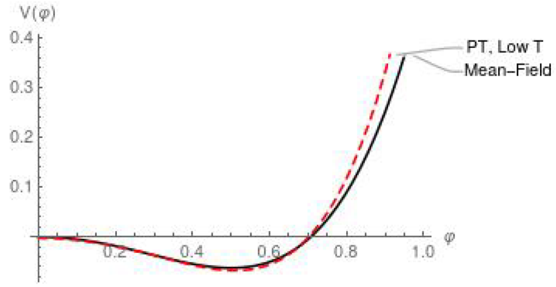


Figure 3.8: Low temperature effective potential curves to zero and first order in perturbation theory.

We can see there is no qualitative change in the potential, and only a very small quantitative change in the curve. If we now consider the high temperature limit ($T \gg m$), we can consider an expansion of J_B in powers of m, which can be gleaned from Eq. (3.45) to give (up to second-order):

$$J_B(T, m) \approx -\frac{\pi^2 T^4}{90} + \frac{T^2}{24} m^2. \tag{3.74}$$

This results in the effective renormalized temperature at high temperatures:

$$U^{\text{ren}}_{\text{eff},0}(\overline{\phi}) = \frac{1}{2}\left(-m^2 + \lambda T^2\right)\overline{\phi}^2 + \lambda\overline{\phi}^4 - \frac{m^4_{\text{eff}}(\overline{\phi})}{64\pi^2}\ln\left(\frac{\bar{\mu}^2 e^{3/2}}{\left|m^2_{\text{eff}}(\overline{\phi})\right|}\right). \tag{3.75}$$

Once again setting $\bar{\mu} = m$ and $\lambda = 1$ gives the plot in Fig. 3.9.

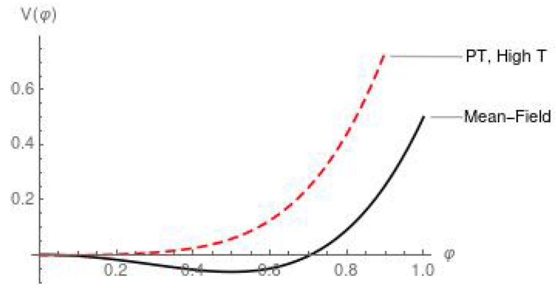


Figure 3.9: High temperature effective potential curves to zero and first order in perturbation theory.

This in fact produces a qualitative change in the behavior, which restores symmetry due to the change in sign of the quadratic term at high temperatures. Even better than showing there is indeed a phase transition, we can also estimate the location at which this phase transition occurs (the transition value of T). To do see, we find the minimum of the potential with respect to $\overline{\phi}$ using the first derivation necessary condition. We know that in the symmetric phase, the minimum must occur at $\overline{\phi} = 0$, so we have

$$\left.\frac{dU^{\text{ren}}_{\text{eff},0}(\overline{\phi})}{d\overline{\phi}}\right|_{\overline{\phi}=0} = m^2\left(-1 + \frac{3\lambda}{4\pi^2}\right) + \lambda T^2 = 0 \tag{3.76}$$

$$\Rightarrow \quad T_c^2 = m^2\left(\frac{1}{\lambda} - \frac{3}{4\pi^2}\right), \tag{3.77}$$

where the subscript "c" stands for critical temperature of the phase transition. Since we are doing perturbation theory and treating $\lambda \ll 1$, we are then left with the approximation of the critical temperature being

$$\boxed{T_c \approx \frac{m}{\sqrt{\lambda}}}. \tag{3.78}$$

§3.5 The Full Propagator

We have seen how to compute the propagator (two-point function, $\langle\phi(x)\phi(y)\rangle$) for a free scalar field in Sec. 3.3. Now, we want to tackle the issue of computing the full propagator of a system of interacting QFTs. The full propagator is given as

$$\begin{aligned} G(\boldsymbol{x}) &= \frac{\int \mathcal{D}_\phi e^{-S_0 - S_I}\phi(\boldsymbol{x})\phi(0)}{Z} \\ &\approx \frac{\int \mathcal{D}_\phi e^{-S_0 - S_I}\phi(\boldsymbol{x})\phi(0)}{Z_{\text{free}}} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\int \mathcal{D}_\phi e^{-S_0}\phi(\boldsymbol{x})\phi(0)\left(-S_I\right)^n}{Z_{\text{free}}}. \end{aligned} \tag{3.79}$$

In this case, the infinite perturbative series for the propagator in terms of (connected) Feynman diagrams will have two external (not closed loops) legs, which in some cases all of the terms can be computed (with diagrams of a simple enough structure). This summing up of an infinite number of terms is known as *resummation* and can in some cases be quite successful an approximation. Let's look at just the first-order correction to the full propagator,

$$G_{(1)}(\boldsymbol{x}) = G_{\text{free}}(\boldsymbol{x}) - \langle\phi(\boldsymbol{x})\phi(0)S_I\rangle. \tag{3.80}$$

Similar, we can also compute the first-order correction to the partition function which will be

$$Z_{(1)} = Z_{\text{free}} - \langle S_I\rangle. \tag{3.81}$$

For to evaluate the first-order corrected propagator, we will first need to evaluate the correction term

$$\begin{aligned} \langle\phi(\boldsymbol{x})\phi(0)S_I\rangle &= \lambda\int d^4y\langle\phi(\boldsymbol{x})\phi(0)\phi^4(\boldsymbol{y})\rangle \\ &= 12\lambda\int d^4y\langle\phi(\boldsymbol{x})\phi(\boldsymbol{y})\rangle\langle\phi(\boldsymbol{y})\phi(0)\rangle\langle\phi(\boldsymbol{y})\phi(\boldsymbol{y})\rangle \\ &= 12\lambda\int d^4y G_{\text{free}}(\boldsymbol{x}-\boldsymbol{y})G_{\text{free}}(\boldsymbol{y})G_{\text{free}}(0), \end{aligned} \tag{3.82}$$

where we used the Wick's theorem above. Now using the delta-function identity

$$\delta(\boldsymbol{P}-\boldsymbol{K}) = \int d^4y e^{i(\boldsymbol{P}-\boldsymbol{K})\boldsymbol{y}}, \tag{3.83}$$

we can assert the zero-temperature limit and compute the first-order correction to get

$$\lim_{T\to 0}\langle\phi(\boldsymbol{x})\phi(0)S_I\rangle = 12\lambda G_{\text{free}}(0)\int d^4K\frac{e^{i\boldsymbol{K}\cdot\boldsymbol{x}}}{(K^2+m^2)^2}, \tag{3.84}$$

$$\Rightarrow \quad \lim_{T\to 0} G_{(1)}(\boldsymbol{x}) = \int d^4K\frac{e^{i\boldsymbol{K}\cdot\boldsymbol{x}}}{K^2+m^2} - 12\lambda G_{\text{free}}(0)\int d^4K\frac{e^{i\boldsymbol{K}\cdot\boldsymbol{x}}}{(K^2+m^2)^2}, \tag{3.85}$$

$$\Rightarrow \quad \tilde{G}_{(1)}(\boldsymbol{K}) = \frac{1}{K^2+m^2} - \frac{12\lambda G_{\text{free}}(0)}{(K^2+m^2)^2}. \tag{3.86}$$

The explicit form of $G_{\text{free}}(0)$ has been computed in Eq. (3.44), and we notice that the result of $\tilde{G}_{(1)}(\boldsymbol{K})$ looks much like the start of a geometric series (which it turns out to be if we compute the higher order corrections explicitly). As such, one finds that the full (resummed) propagator in momentum space is given by

$$\tilde{G}(\boldsymbol{K}) = \frac{1}{K^2+m^2+12\lambda G_{\text{free}}(0)} + \mathcal{O}(\lambda^2), \tag{3.87}$$

which is valid up to second-order in perturbation theory. We can fix this result to be exact if we replace $12\lambda G_{\text{free}}(0)$ in the denominator with the appropriate function of $\boldsymbol{K}$ we call $\tilde{\Pi}(\boldsymbol{K})$, such that

$$\boxed{\tilde{G}(\boldsymbol{K}) = \frac{1}{K^2+m^2+\tilde{\Pi}(\boldsymbol{K})}}. \tag{3.88}$$

The function Π in real-space is referred to as the *self-energy* of the scalar field ϕ, which is a central object in QFT (contains a lot of information) and something we will return to in generality later in the course. For now, let's just look at the lowest order term of $\tilde{\Pi}(\boldsymbol{K})$, which we can get by simply plugging in Eq. (3.44) to give

$$\begin{aligned}\tilde{\Pi}_{(1)} &= 12\lambda G_{\text{free}}(0)\\ &= -\frac{3m^2\lambda}{4\pi^2}\left[\frac{1}{\varepsilon}+\ln\left(\frac{\bar{\mu}^2e^{\frac{1}{2}}}{m^2}\right)\right]+12\lambda I_B(T,m).\end{aligned} \tag{3.89}$$

Unfortunately, this result is divergent when $\varepsilon\to 0$. However, since $\tilde{\Pi}_{(1)}(\boldsymbol{K})$ always appears in conjunction with m^2 (as seen in Eq. 3.88), we can use m which is a parameter of the Lagrangian to renormalize this (i.e. add a counter term) by taking

$$m^2 \quad\to\quad m^2_{\text{phys}}+\delta m^2, \tag{3.90}$$

$$\text{with}\quad \delta m^2 \equiv \frac{2m^2_{\text{phys}}\lambda}{4\pi^2\varepsilon}+\mathcal{O}(\lambda^2). \tag{3.91}$$

Note: The subscript "phys" does **not** imply that the quantity is physically measurable, but rather associated to the renormalization scheme to renormalize the mass parameter to something physical. This notation will be adopted for other such renormalization counter terms in future.

As a result, we are left with a finite result

$$\begin{aligned} m^2 + \tilde{\Pi}_{(1)} &= m_{\text{phys}}^2 - \frac{3m_{\text{phys}}^2\lambda}{4\pi^2}\ln\left(\frac{\bar{\mu}^2 e^{\frac{1}{2}}}{m_{\text{phys}}^2}\right) + 12\lambda I_B(T, m_{\text{phys}}) \\ &= m_{\text{phys}}^2 + \tilde{\Pi}_{(1)}^{\text{ren}} + \mathcal{O}(\lambda^2). \end{aligned} \tag{3.92}$$

Recall that in Sec. 3.3.1, we found that the poles of the analytically continued free propagator correspond to quasiparticle masses. As such, we can apply this same exercise to the full propagator, which when we replace the Euclidean 4-momentum with the Minkowski one ($K^2 \to -k_0^2 + k^2$), we get a propagator pole at

$$k_0^2 = k^2 + m_{\text{phys}}^2 + \tilde{\Pi}_{(1)}^{\text{ren}}. \tag{3.93}$$

This means that the quasiparticle mass m_{eff}, is then given by

$$m_{\text{eff}}(T) = m_{\text{phys}} - \frac{3\lambda m_{\text{phys}}}{8\pi^2}\ln\left(\frac{\bar{\mu}^2 e^{\frac{1}{2}}}{m_{\text{phys}}^2}\right) + \frac{6\lambda}{m_{\text{phys}}} I_B\left(T, m_{\text{phys}}\right), \tag{3.94}$$

which seems to be renormalization scale and temperature dependent. However since it is a measurable quantity, this cannot be the case. The only way out is if

$$\bar{\mu}\frac{\partial m_{\text{eff}}^2}{\partial \bar{\mu}} = 0. \tag{3.95}$$

For this to hold, it must then be that m_{phys} is dependent on the scale $\bar{\mu}$ for which we say that the mass parameter "runs". Plugging this into the condition above and taking the $T = 0$ limit gives

$$\bar{\mu}\frac{\partial m_{\text{phys}}^2(\bar{\mu})}{\partial \bar{\mu}}\left[1 - \frac{3\lambda}{4\pi^2}\ln\left(\frac{\bar{\mu}^2 e^{-\frac{1}{2}}}{m_{\text{phys}}^2(\bar{\mu})}\right)\right] - \frac{3\lambda m_{\text{phys}}^2(\bar{\mu})}{2\pi^2} = 0, \tag{3.96}$$

for which asserting perturbation theory and dropping all λ^2 terms and higher gives

$$\boxed{\bar{\mu}\frac{\partial m_{\text{phys}}^2(\bar{\mu})}{\partial \bar{\mu}} = \frac{3\lambda m_{\text{phys}}^2(\bar{\mu})}{2\pi^2}}. \tag{3.97}$$

A special case would be when $m_{\text{phys}} = 0$, which would result in the effective mass (with non-zero temperature) to take the form

$$m_{\text{eff}}(T) = \sqrt{12\lambda I_B(T, 0)} = T\sqrt{\lambda}. \tag{3.98}$$

This turns out to be an actual physically realizable scenario known as the *in-medium mass*, where massless quasiparticles acquire an effective mass through interactions with the thermal medium.

> **Note**: The in-medium mass is generically unavoidable for any value of m_{phys}, just that is takes the particularly simple form for the $m_{\text{phys}} = 0$ case.

§3.6 Four-Point Functions

So far we have been studying the formalism of two-point functions (propagators), and so the natural progression from this is to look at four-point functions (FPF):

$$\Gamma_4(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4) = \frac{\int \mathcal{D}_\phi e^{-S_0 - S_I} \phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\phi(\boldsymbol{x}_3)\phi(\boldsymbol{x}_4)}{Z}, \tag{3.99}$$

which are the next order correction terms that appear in the perturbative series. As with the propagator, we only care about **connected** four-point functions since all the disconnected terms will cancel out. The minimum number of vertices to connect 4 fields would be 1 vertex, so the leading order perturbative term would be

$$\begin{aligned}
\Gamma^{(1)}_{4,\text{conn.}} &= \frac{\int \mathcal{D}_\phi e^{-S_0}(-S_I)\phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\phi(\boldsymbol{x}_3)\phi(\boldsymbol{x}_4)}{Z_{(1)}} \\
&= -4!\lambda \int d^4x G_{\text{free}}(\boldsymbol{x}-\boldsymbol{x}_1)G_{\text{free}}(\boldsymbol{x}-\boldsymbol{x}_2)G_{\text{free}}(\boldsymbol{x}-\boldsymbol{x}_3)G_{\text{free}}(\boldsymbol{x}-\boldsymbol{x}_4) \\
&= -4!\lambda \int_{P_1,P_2,P_3,P_4} \tilde{G}_{\text{free}}\,(\boldsymbol{P}_1)\,\tilde{G}_{\text{free}}\,(\boldsymbol{P}_2)\,\tilde{G}_{\text{free}}\,(\boldsymbol{P}_3)\,\tilde{G}_{\text{free}}\,(\boldsymbol{P}_4) \\
&\qquad \times (2\pi)^4 \delta\left(\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3 + \boldsymbol{P}_4\right) e^{i\boldsymbol{P}_1\cdot x_1 + i\boldsymbol{P}_2\cdot x_2 + i\boldsymbol{P}_3\cdot x_3 + i\boldsymbol{P}_4\cdot x_4},
\end{aligned} \tag{3.100}$$

where we once again used Wick's theorem. The Dirac-delta function simply ensures that 4-momentum is conserved. We can then also denote this as an "*amputated*" FPF by dropping the 4 Greens functions since they are a trivial consequence of having 4 external legs, and the delta function with its normalization, which results in the Fourier transformed amputated FPF

$$\tilde{\Gamma}^{(1)}_4\Big|_{\text{conn., amp.}} (\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3 + \boldsymbol{P}_4) = -4!\lambda. \tag{3.101}$$

> **Note**: The "amputated" function is purely a notational device (so we can be lazy and write less terms) which does not actually consider the differently defined function.

We can now also consider the next-to-leading order (NLO) term in the perturbation series which is written as

$$\Gamma^{(2)}_{4,\text{conn.}} = \Gamma^{(1)}_{4,\text{conn.}} + \frac{1}{2}\frac{\int \mathcal{D}_\phi e^{-S_0} \int_{x,y} \phi^4(\boldsymbol{x})\phi^4(\boldsymbol{y})\phi(\boldsymbol{x}_1)\phi(\boldsymbol{x}_2)\phi(\boldsymbol{x}_3)\phi(\boldsymbol{x}_4)}{Z_{(2)}} \tag{3.102}$$

The overall multiplicative factor from this term is given by (excluding the factor of 1/2 out front)

$$(2 \times 4) \times 3 \times (4 \times 3) \times 2 = 576, \tag{3.103}$$

which comes from the following connections:

1. First consider $\phi(x_1)$: there are (2×4) legs in $S_I^2 \sim \phi^4(\boldsymbol{x})\phi^4(\boldsymbol{y})$ to attach to, for which we choose $\phi(x)$;
2. Next, consider $\phi(x_2)$: there are 3 remaining $\phi(x)$ legs to attach to if we want $\phi(x_2)$ connected to the same vertex as $\phi(x_1)$;
3. Next, consider one of the remaining $\phi(x)$: attach this to one of the 4 $\phi(y)$ legs, and then pick the last $\phi(x)$ and attach to the 3 $\phi(y)$;
4. Finally: attach $\phi(x_3)$ to one of the 2 remaining $\phi(y)$ and $\phi(x_4)$ to the last $\phi(y)$.

We can now once again define the amputated FPF in momentum space after performing all contractions (connections of legs) to get

$$\tilde{\Gamma}^{(2)}_{4,\text{conn., amp.}} = \tilde{\Gamma}^{(1)}_{4,\text{conn., amp.}} + \frac{(4!\lambda)^2}{2} \int_K \frac{1}{K^2 + m^2} \frac{1}{(P_1 + P_2 - K)^2 + m^2} + \text{two others}, \tag{3.104}$$

where the two other terms arise from the other options of attaching $\phi(x_2)$, which give rise to the same amputated four-point function but with variables x_1, x_2, x_3, x_4 exchanged. Unlike the the first order term, we have that the amputated function now depends on the momentum of the incoming particles. However, we can consider the special case where all of these momenta are set to zero (zero-temperature limit), which gives

$$\tilde{\Gamma}^{(2)}_{4,\text{conn., amp.}}(P = 0) = (-4!\lambda) + \frac{3(4!\lambda)^2}{2} \int_K \frac{1}{(K^2 + m^2)^2}. \tag{3.105}$$

We have evaluated this integral before in Eq. (2.28), which gives us that

$$\begin{aligned} \int_K \frac{1}{(K^2 + m^2)^2} &= \frac{1}{(4\pi)^{\frac{4}{2}}} \frac{\Gamma(2 - \frac{4}{2})}{\Gamma(2)} (m^2)^{-2+\frac{4}{2}} \\ &= \frac{1}{(4\pi)^2} \frac{\Gamma(0)}{\Gamma(2)}. \end{aligned} \tag{3.106}$$

This gives a divergent result, so we need to use dimensional regularization and consider $D = 4 - 2\varepsilon$, such that we instead are left with

$$\tilde{\Gamma}^{(2)}_{4,\text{conn., amp.}}(P = 0) = (-4!\lambda) + \frac{3(4!\lambda)^2}{2} \frac{1}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon) \left(\frac{\mu^2}{m^2}\right)^{\varepsilon}. \tag{3.107}$$

Expanding in powers of ε to lowest order gives

$$\tilde{\Gamma}^{(2)}_{4,\text{conn., amp.}}(P = 0) = (-4!\lambda) + \frac{3(4!\lambda)^2}{32\pi^2} \left[\frac{1}{\varepsilon} + \ln\left(\frac{\overline{\mu}^2}{m^2}\right)\right] + \mathcal{O}(\varepsilon). \tag{3.108}$$

To now renormalize the theory, we use the coupling constant as the renormalization parameter, completely analogous to a renormalization of the mass parameter in Eq. (3.90), such that

$$\lambda \quad \to \quad \lambda_{\text{phys}} + \delta\lambda, \tag{3.109}$$

where $\delta\lambda = \mathcal{O}(\lambda_{\text{phys}}^2)$. In the $\overline{\text{MS}}$ scheme, we would then choose

$$\delta\lambda = \frac{9\lambda_{\text{phys}}^2}{4\pi^2\varepsilon} \tag{3.110}$$

$$\Rightarrow \quad \tilde{\Gamma}^{(2)}_{4,\text{conn., amp., ren.}}(P=0) = (-4!\lambda_{\text{phys}}) + \frac{9\lambda_{\text{phys}}^2}{4\pi^2}\ln\left(\frac{\bar{\mu}^2}{m^2}\right). \tag{3.111}$$

We are left with one loose end to tie still, and that is since $\tilde{\Gamma}^{(2)}_{4,\text{conn., amp.}}$ is in principle a measurable quantity, it should not be $\overline{\mu}$ scale dependent. We fix this by asserting a condition similar to Eq. (3.95), which translates to this context as

$$\overline{\mu}\frac{\partial\tilde{\Gamma}^{(2)}_{4,\text{conn., amp., ren.}}(P=0)}{\partial\overline{\mu}} = 0. \tag{3.112}$$

This condition can only be met if λ_{phys} is itself $\overline{\mu}$ dependent, which grants the differential equation

$$\bar{\mu}\frac{\partial\lambda_{\text{phys}}(\bar{\mu})}{\partial\bar{\mu}}\left[1 - \frac{3\lambda_{\text{phys}}}{16\pi^2}\ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right] - \frac{3\lambda_{\text{phys}}^2}{16\pi^2} = 0, \tag{3.113}$$

which to leading order in perturbation theory (powers of λ_{phys}) simplifies to

$$\boxed{\bar{\mu}\frac{\partial\lambda_{\text{phys}}(\bar{\mu})}{\partial\bar{\mu}} = \frac{3\lambda_{\text{phys}}^2}{16\pi^2} + \mathcal{O}(\lambda_{\text{phys}}^3)}. \tag{3.114}$$

As per the "lingo" we used earlier, we say that the coupling constant "runs".

§3.7 Renormalization Group

We have seen that in at several instances in perturbative QFT, we have come up against physical observables in the theory that somehow depend on a scale parameter $\overline{\mu}$, after renormalization. This should **not** be the case for an actual theory, although the Lagrangian **can** depend on $\overline{\mu}$. Specifically, we saw this arise when attempting to compute the four-point function which would be necessary as a second-order correction to the path integral. Independence of the quasiparticle mass led to Eq. (3.97):

$$\bar{\mu}\frac{\partial m_{\text{phys}}^2(\bar{\mu})}{\partial\bar{\mu}} \approx \frac{3\lambda m_{\text{phys}}^2(\bar{\mu})}{2\pi^2}, \tag{3.115}$$

while independence of the physical vertex led to Eq. (3.114):

$$\bar{\mu}\frac{\partial\lambda_{\text{phys}}(\bar{\mu})}{\partial\bar{\mu}} \approx \frac{3\lambda_{\text{phys}}^2}{16\pi^2}. \tag{3.116}$$

This implied a scale dependence of m_{phys} and λ_{phys} known as the running mass and coupling constant respectively. Before proceeding, it is customary to introduce the notation

$$\gamma_m(\lambda) \equiv \bar{\mu}\frac{\partial \ln\left[m^2(\bar{\mu})\right]}{\partial\bar{\mu}}, \tag{3.117a}$$

$$\beta(\lambda) \equiv \bar{\mu}\frac{\partial\lambda(\bar{\mu})}{\partial\bar{\mu}}. \tag{3.117b}$$

To handle this issue with a formal scheme, Richard Feynman, Julian Schwinger and Shinichiro Tomonaga came up with the *renormalization group* formalism (the group of scale transformations). To best way to learn about this concept is by first considering a physically measurable object such as the pressure. We have seen that calculating pressures in QFT typically require renormalization and thus a dependence on Lagrangian parameters, i.e.:

$$p_{\text{ren}} = p_{\text{ren}}\left(\bar{\mu}, \lambda_{\text{phys}}(\bar{\mu}), m_{\text{phys}}(\bar{\mu})\right). \tag{3.118}$$

We consider physical observables because we know for certain that these quantities **must** be scale invariant (i.e. $p_{\text{ren}}(\overline{\mu}) = p_{\text{ren}}(\overline{\mu}'))$, which is referred to as "*renormalization group (RG) invariance*". For pressure, this condition can be written as

$$\overline{\mu}\frac{d}{d\overline{\mu}}p_{\text{ren}}\left(\overline{\mu}, \lambda_{\text{phys}}(\overline{\mu}), m_{\text{phys}}(\overline{\mu})\right) = 0 \tag{3.119}$$

$$\Rightarrow \quad \left[\bar{\mu}\frac{\partial}{\partial\bar{\mu}} + \beta\frac{\partial}{\partial\lambda_{\text{phys}}} + \gamma_m m^2_{\text{phys}}\frac{\partial}{\partial m^2_{\text{phys}}}\right]p_{\text{ren}}\ \left(\bar{\mu}, \lambda_{\text{phys}}\ , m_{\text{phys}}\ \right) = 0. \tag{3.120}$$

It turns out that calculations are easier for another related physical quantity, the entropy density $s = \partial p/\partial T$ as it will avoid having to deal with issues regarding the cosmological constant. Now using Eq. (2.42) and Eq. (3.47) which read,

$$p^{\text{renorm}}(T) = -J_B(T,m) + \frac{m^4}{64\pi^2}\ln\left(\frac{\overline{\mu}^2 e^{3/2}}{m^2}\right), \tag{3.121}$$

$$p_{(1)} = p_{\text{free}} - 3\lambda G^2_{\text{free}}(0), \tag{3.122}$$

where $G_{\text{free}}(0)$ is given by Eq. (3.44), we can compute the entropy density up to first-order in perturbation theory as

$$s_{(1)} = -\frac{\partial}{\partial T}J_B(T,m) - 6\lambda G_{\text{free}}(0)\frac{\partial}{\partial T}I_B(T,m), \tag{3.123}$$

where $I_B(T,m)$ is defined in Eq. (3.45). Since J_B and I_B are not functions of $\overline{\mu}$ but $G_{\text{free}}(0)$ is, we get

$$\overline{\mu}\frac{\partial}{\partial\overline{\mu}}s_{(1)} = -6\lambda\overline{\mu}\frac{\partial G_{\text{free}}(0)}{\partial\overline{\mu}}\frac{\partial I_B(T,m)}{\partial T} \tag{3.124}$$

$$\Rightarrow \quad 6\lambda\frac{\partial I_B(T,m)}{\partial T}\frac{m^2}{8\pi^2} = -\left[\beta\frac{\partial}{\partial\lambda_{\text{phys}}} + \gamma_m m^2_{\text{phys}}\frac{\partial}{\partial m^2_{\text{phys}}}\right]s_{(1)}, \tag{3.125}$$

where we plugged in the explicit form of $G_{\text{free}}(0)$ and used the RG invariance criteria

$$\begin{aligned} \left[\bar{\mu}\frac{\partial}{\partial\bar{\mu}} + \beta\frac{\partial}{\partial\lambda_{\text{phys}}} + \gamma_m m_{\text{phys}}^2 \frac{\partial}{\partial m_{\text{phys}}^2}\right] s_{(1)} = 0, \\ \Rightarrow \quad \bar{\mu}\frac{\partial}{\partial\bar{\mu}} s_{(1)} = -\left[\beta\frac{\partial}{\partial\lambda_{\text{phys}}} + \gamma_m m_{\text{phys}}^2 \frac{\partial}{\partial m_{\text{phys}}^2}\right] s_{(1)}. \end{aligned} \tag{3.126}$$

To lowest order in perturbation theory, we have that $m = m_{\text{phys}}$ and $\lambda = \lambda_{\text{phys}}$. Also, β by definition is of $\mathcal{O}(\lambda^2)$, which leaves

$$\begin{aligned} \frac{\partial I_B}{\partial T}\frac{3\lambda m^2}{4\pi^2} &\approx -\gamma_m m_{\text{phys}}^2 \frac{\partial}{\partial m_{\text{phys}}^2} s_{(1)} \\ &= \gamma_m m_{\text{phys}}^2 \frac{\partial}{\partial m_{\text{phys}}^2}\frac{\partial J_B}{\partial T} \\ &= \gamma_m \frac{m_{\text{phys}}^2}{2}\frac{\partial I_B}{\partial T} \\ &= \left(\frac{3\lambda_{\text{phys}}}{2\pi^2}\right)\frac{m_{\text{phys}}^2}{2}\frac{\partial I_B}{\partial T}. \end{aligned} \tag{3.127}$$

So indeed, we see that to lowest order the RG invariance criteria is met.

We see from this example that RG is useful for consistency checks in perturbation theory, but more importantly imply evolution equations (such as those in Eq. (3.97) and Eq. (3.114)) which can be used to solve for Lagrangian parameters. For illustration, we shall solve for the form of the running coupling λ_{phys}, in ϕ^4 theory to leading order in perturbation theory from Eq. (3.114) as follows:

$$\begin{aligned} \bar{\mu}\frac{\partial\lambda_{\text{phys}}(\bar{\mu})}{\partial\bar{\mu}} &\approx \frac{3\lambda_{\text{phys}}^2}{16\pi^2}, \\ \Rightarrow \quad \frac{1}{\lambda_{\text{phys}}^2}\frac{\partial\lambda_{\text{phys}}(\bar{\mu})}{\partial\bar{\mu}} &\approx \frac{3\bar{\mu}}{16\pi^2} \\ \Rightarrow \quad -\frac{\partial\lambda_{\text{phys}}^{-1}(\overline{\mu})}{\partial\ln\overline{\mu}} &\approx \frac{3}{16\pi^2} \\ \Rightarrow \quad \boxed{\lambda_{\text{phys}}(\overline{\mu}) = \frac{16\pi^2/3}{\ln(\mu_0/\overline{\mu})}}&, \end{aligned} \tag{3.128}$$

where μ_0 is an integration constant with mass dimension 1. This can be plot as shown in Fig. 3.10

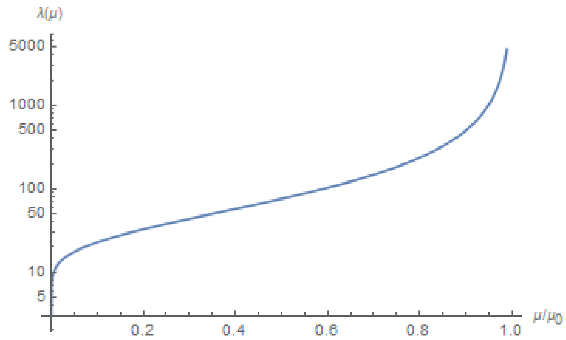


Figure 3.10: Plot of $\lambda(\overline{\mu})$ vs $\overline{\mu}/\mu_0$ for first-order ϕ^4 theory.

So we see that $\lambda(\overline{\mu})$ is small for small energy scales (shorted distance scales) $\overline{\mu}$, and grows monotonically as a function of $\overline{\mu}$ (running). This implies that the coupling get stronger at short distances which would cause problems to arise in the *continuum limit* of the field theory.

Note: The monotonic increase is due to the positive sign associated to the β function defined earlier. If it were instead negative, the trend would be decreasing with $\overline{\mu}$, implying weaker and weaker coupling at shorter distances which are referred to *asymptotically free theories.*

A perculiar result for ϕ^4 theories is that when $\overline{\mu} = \mu_0$, the coupling diverges. this value of $\mu_0 = \overline{\mu}$ is called the *Landau pole.* Although this result is questionable since our result derives from perturbation theory in which the coupling is assumed to be small. However in more general cases where Landau poles do certainly exist, it implies a minimum length scale $\propto \mu_0^{-1}$ below which the theory fails (i.e. the theory is a cut-off dependent effective field theory).

§3.8 Complex Scalar Fields

Thus far, we have been working under the assumption of a real valued scalar field which due to Lorentz invarinace, requires the Euclidean action of the form

$$S_E = \int d^4x_E \left[\frac{1}{2}\partial_a\phi\partial_a\phi + V(\phi)\right]. \tag{3.129}$$

We are now going to extend our theories to encompass *complex* scalar fields (these are **not** observed in nature). To make sure that this still results in physically acceptable QFTs, we must ensure that physical observables remain real and we expect the partition function $Z \in \mathbb{R}$. The simplest way to ensure $Z \in \mathbb{R}$ is to work with a real-valued action. Real actions can be constructed out of complex scalar fields by using quadratic form terms such as $\phi\phi^*$. Once again asserting Lorentz invariance, we can write down the action

$$S_E = \int d^4x_E \left[\partial_a\phi\partial_a\phi^* + V(\sqrt{\phi\phi^*})\right], \tag{3.130}$$

which satisfies all the necessary conditions so far. For starters, it is instructive to consider a complex scalar field QFT which is close to those we have studied for real scalar fields. As such, we write the following action:

$$S_E = \int d^4x_E \left[\partial_a\phi\partial_a\phi^* + m^2\phi\phi^* + 4\lambda(\phi\phi^*)^2\right]. \tag{3.131}$$

With the introduction of complex fields, we note that the action (apart from Lorentz invariance) has an additional symmetry:

$$\phi(x) \;\to\; e^{i\alpha}\phi(x) \quad \text{or} \quad \phi^*(x) \;\to\; e^{-i\alpha}\phi^*(x), \tag{3.132}$$

where $\alpha \in \mathbb{R}$ is a constant. In group theory, this is known as a $U(1)$ *transformation*. We will come back to the implications of this symmetry soon, but for now we shall work at computing the resulting partition function for this theory. Since ϕ is complex, we can separate it into =real and imaginary components written as

$$\phi(x) = \frac{\phi_1(x) + i\phi_2(x)}{\sqrt{2}}, \tag{3.133}$$

$$\Rightarrow \quad S_E = \int d^4x_E \left[\frac{1}{2}\partial_a\phi_1\partial_a\phi_1 + \frac{1}{2}\partial_a\phi_2\partial_a\phi_2 + \frac{m^2}{2}\left(\phi_1^2+\phi_2^2\right) + \lambda\left(\phi_1^2+\phi_2^2\right)^2\right]. \tag{3.134}$$

The partition function is then a path integral over both ϕ_1 and ϕ_2 written as

$$Z = \int \mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2}e^{-S_E}, \tag{3.135}$$

which looks very much like 2 copies of the partition function for a real scalar field. There is however one term which mediates the 2 copies which is the cross term $2\lambda\phi_1^2\phi_2^2$. To simplify our discussion, we shall consider the case where we drop the coupling ($\lambda = 0$) for the time being. In this case, we just have the free complex scalar field which we can decompose as

$$S_E\Big|_{\lambda=0} = S_0[\phi_1] + S_1[\phi_2], \tag{3.136a}$$

$$\text{where} \quad S_0[\phi] = \int d^4x_E \left[\frac{1}{2}\partial_a\phi_{\mathbb{R}}\partial_a\phi_{\mathbb{R}} + \frac{m^2}{2}\phi_{\mathbb{R}}^2\right], \tag{3.136b}$$

with $\phi_{\mathbb{R}}$ is a real scalar field. So the free partition function for the complex scalar field is now just

$$\begin{aligned} Z_{\text{free}} &= \int \mathcal{D}_{\phi_1}\mathcal{D}_{\phi_2}e^{-S_0[\phi_1]-S_1[\phi_2]} \\ &= \int \mathcal{D}_{\phi_1}e^{-S_0[\phi_1]}\int \mathcal{D}_{\phi_2}e^{-S_1[\phi_2]} \\ &= \left[Z_{\text{free}}(\phi_{\mathbb{R}})\right]^2. \end{aligned} \tag{3.137}$$

The free pressure for a complex scalar field would then be

$$\begin{aligned} p^{\mathbb{C}}_{\text{free}} &= \frac{\ln Z_{\mathbb{C}}}{\beta V} \\ &= \frac{\ln(Z_{\mathbb{R}})^2}{\beta V} = \frac{2\ln Z_{\mathbb{R}}}{\beta V}. \end{aligned} \tag{3.138}$$

In the zero mass limit ($m = 0$), we can use the result from before and compute that zero-mass free pressure for a complex scalar field as

$$\boxed{\lim_{m\to 0} p^{\mathbb{C}}_{\text{free}} = \frac{\pi^2 T^4}{45}}. \tag{3.139}$$

The physics behind this is rather simple here since as we already saw, the pressure for a complex scalar field just corresponds to the pressure due to two real scalar fields. Generalizing this, we see that the pressure due to N free scalar fields would just be

$$p^{(N)}_{\text{free}} = \frac{N\pi^2 T^4}{90}. \tag{3.140}$$

We call every term of $\pi^2 T^4/90$ as a *bosonic degree of freedom* (bosonic in reference to the field being scalar). This generalization in fact allows one to count the number of bosonic degrees of freedom in the system via the formula

$$\text{DoF} = \frac{90p(T)}{\pi^2 T^4}. \tag{3.141}$$

The DoF defined in this way does **not** need to be an integer, and works as an effective number of degrees of freedom. This concept of effective DoF is relevant for areas such as cosmology where the DoF in fact changes in time, which is a means for cosmologist to characterize the history of the universe.

§3.8.1 Noether's Theorem: Symmetries and Conserved Quantities

Symmetries are an integral part of physics in general, for which is connection to conservation laws was made precise by the groundbreaking work of Emmy Noether in 1915. In this section, we will explore a particular symmetry alluded earlier where a complex scalar field picks up an arbitrary phase $e^{i\alpha}$. First a general treatment, taking ϕ and ϕ^* as independent fields we can compute the variation of the action as

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) + \frac{\partial \mathcal{L}}{\partial \phi^*}\delta\phi^* + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\delta(\partial_\mu\phi^*)\right]. \tag{3.142}$$

To compute this, we first consider the chain rule (i.e. integration by parts) which gives

$$\int d^4x \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \int d^4x\, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)\delta\phi}\right] - \int d^4x \delta\phi\, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right], \tag{3.143}$$

and similarly for ϕ^*. This results in the equations of motion for ϕ and ϕ^* as

$$(\Box - m^2)\phi + 8\lambda\,(\phi\phi^*)\,\phi, \tag{3.144a}$$

$$(\Box - m^2)\phi^* + 8\lambda\,(\phi\phi^*)\,\phi^*, \tag{3.144b}$$

where $\Box$ is the *d'Alembertian.* Using these equations of motion and applying them back to the variation in the action gives

$$\delta S = \int d^4x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^*\right], \tag{3.145}$$

which vanishes if the function in the total derivative is well behaved at infinity ($\delta S = 0$) for any small variation $\delta\phi$. If we now consider the system placed in a box of finite volume such that, assuming the variation $\delta\phi$ is an actual symmetry of the system, we will still retain that

$$\delta S_{\text{box}} = \int_{\text{box}} d^4x \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^* \right] = 0. \tag{3.146}$$

Now going back to our symmetry of interest, if we consider the variation

$$\begin{aligned} \delta\phi &= \phi - \phi' \\ &= \phi - e^{i\alpha}\phi \approx i\alpha\phi, \end{aligned} \tag{3.147}$$

this will leads to

$$\begin{aligned} \alpha \int_{\text{box}} d^4x \partial_\mu \left[i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi - i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \phi^* \right] &= 0 \\ \Rightarrow \quad \partial_\mu \left[i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi - i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \phi^* \right] &= 0, \end{aligned} \tag{3.148}$$

where the integrand must vanish if the integral does since the box is an arbitrary volume. We denote the contravariant term in brackets as

$$\begin{aligned} &\boxed{j^\mu(x) = i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi - i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \phi^*}, \\ \Rightarrow \quad &j^\mu(x) = 2 \, \text{Im} \left\{ \phi \partial_\mu \phi^* \right\}, \end{aligned} \tag{3.149}$$

where the second line comes from plugging in the explicit form of the Lagrangian into the expression for $j^\mu(x)$. This object is referred to as the *Noether current density*, which we see is conserved if $\delta\phi$ is indeed a symmetry of the system:

$$\boxed{\partial_\mu j^\mu(x) = 0}. \tag{3.150}$$

Separating out the space and time components, we get the relation

$$\partial_\mu j^\mu(x) = \partial_0 \rho + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0, \tag{3.151}$$

which if integrated over an infinite spatial volume ($\int d^3x$), grants the relation

$$\partial_0 \int d^3x \rho = 0, \tag{3.152}$$

as long as $\boldsymbol{j}$ is well behaved and falls off sufficiently fast at infinity. We call the integral $\int d^3x\rho$ the *Noether charge* Q, which is conserved in time $\partial_0 Q = 0$. These results constitute what is known as *Noether's theorem*, which reads:

Theorem 3.8.1. *Every continuous symmetry corresponds to a conserved quantity.*

This of course extends to all symmetries (e.g. time translation symmetry leading to energy conservation) and not just the symmetry we are studying in this section, making it one of the most important results in modern physics.

§3.8.2 Noether's Theorem: Quantum Mechanics

In the previous section, we saw how conserved quantities arise in classical fields in the presence of continuous symmetries. In this section, we will extend this concept to tackle quantum mechanical systems. This will require us to ensure that the symmetries we saw in the action extend also to the path integral (elevating the classical action to a full quantum theory). It turns out that working in terms of the Minkowski action (by performing the Wick rotation) is easier for this task, so we write the partition function written as

$$Z = \int \mathcal{D}_\phi e^{iS}, \tag{3.153}$$

$$\text{where} \quad S = -\int d^4x \left[\partial_\mu \phi \partial^\mu \phi^* + m^2|\phi|^2 + 4\lambda|\phi|^4\right]. \tag{3.154}$$

The path integral for complex scalar fields will then have to be

$$Z = \int \mathcal{D}_\phi \mathcal{D}_{\phi^*} e^{iS[\phi,\phi^*]}. \tag{3.155}$$

In order to see if the symmetry of $\phi \to \phi' = e^{i\alpha}\phi$ is indeed a symmetry of the QFT, we will have to check if

$$\int \mathcal{D}_\phi \mathcal{D}_{\phi^*} e^{iS[\phi,\phi^*]} = \int \mathcal{D}_{\phi'} \mathcal{D}_{\phi'^*} e^{iS[\phi',\phi'^*]}. \tag{3.156}$$

Before doing so, we will make another generalization to the transformation we had earlier. That is, we will now make $\alpha = \alpha(x)$, such that the transformation now become *locally defined*:

$$\phi(x) \quad \to \quad e^{i\alpha(x)}\phi(x). \tag{3.157}$$

Since the result of Noether's theorem holds for any small variation $\delta\phi(x)$ at least to linear order, it stands to reason that the transformation above would also hold granted that $\alpha(x)$ is small such that the variation is instead

$$\delta\phi(x) = i\alpha(x)\phi(x) \tag{3.158}$$

$$\begin{aligned}
\Rightarrow \quad \delta S &= \int d^4x \partial_\mu \left[\alpha(x)\left(i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi - i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\phi^*\right)\right] \\
&= \int d^4x\, \partial_\mu \left[\alpha(x) j^\mu(x)\right] \\
&= \int d^4x\, \alpha(x)\partial_\mu j^\mu(x) + \int d^4x\, j^\mu(x)\partial_\mu \alpha(x) \\
&= \int d^4x\, j^\mu(x)\partial_\mu \alpha(x),
\end{aligned} \tag{3.159}$$

since we know that $\partial_\mu j^\mu(x) = 0$. With this, we can write the path amplitude as

$$\begin{aligned}
e^{iS[\phi',\phi'^*]} &= e^{iS[\phi,\phi^*]}e^{i\delta S} \\
&= e^{iS[\phi,\phi^*]}\left(1 + i\int d^4x j^\mu \partial_\mu \alpha\right).
\end{aligned} \tag{3.160}$$

The path integral measure under the transformation becomes

$$\mathcal{D}_\phi \quad \to \quad \mathcal{D}_\phi e^{i\alpha(x)}, \tag{3.161a}$$

$$\mathcal{D}_{\phi^*} \quad \to \quad \mathcal{D}_{\phi^*} e^{-i\alpha(x)}, \tag{3.161b}$$

which implies that $\mathcal{D}_\phi \mathcal{D}_{\phi^*} = \mathcal{D}_{\phi'} \mathcal{D}_{\phi'^*}$. As such, we have that to linear order in the parameter $\alpha(x)$:

$$\begin{aligned} & \int d^4x \partial_\mu \alpha(x) \int \mathcal{D}_\phi \mathcal{D}_{\phi^*} e^{iS} j^\mu(x) = 0, \\ \Rightarrow \quad & \int d^4x \partial_\mu \alpha(x) \times \langle j^\mu(x) \rangle_{\text{full}} = 0, \\ \Rightarrow \quad & \int d^4x \alpha(x) \partial_\mu \langle j^\mu(x) \rangle_{\text{full}} = 0, \\ \Rightarrow \quad & \boxed{\partial_\mu \langle j^\mu(x) \rangle_{\text{full}} = 0}, \end{aligned} \tag{3.162}$$

where we noted that the integrand must vanish along with the integral since $\alpha(x)$ although small, is an arbitrary function of x. So we see that in the full QFT, the **expectation value of the Noether current is conserved**.

§3.8.3 Ward-Takahashi Identity

In QFT, alongside the Noether conservation laws is an additional constraint on correlation functions which arise from local transformation invariance. This additional constraint is known as the *Ward-Takahashi identity*, and are powerful because they provide an exact (nonperturbative) relation in QFT. To start off, we consider the two-point function for a complex scalar field

$$\langle \phi(x) \phi^*(y) \rangle_{\text{full}} = \frac{\int \mathcal{D}_\phi \mathcal{D}_{\phi^*} e^{iS} \phi(x) \phi^*(y)}{Z}. \tag{3.163}$$

Considering once again the local transformation, we see that since it leaves the action invariant (to first-order in α), along with the path integral measure and quadratic field terms invariant, we have that

$$\int \mathcal{D}_\phi \mathcal{D}_{\phi^*} e^{iS} \phi(x) \phi^*(y) = \int \mathcal{D}_{\phi'} \mathcal{D}_{\phi'^*} e^{iS'} \phi'(x) \phi'^*(y). \tag{3.164}$$

However, if we expand the quadratic field term and path amplitude (done in the previous section) in the integral up to linear order in α, we get

$$\phi'(x) \phi'^*(y) = \phi(x) \phi^*(y) \left[1 + i\alpha(x) - i\alpha(y)\right], \tag{3.165}$$

$$e^{iS[\phi', \phi'^*]} = e^{iS[\phi, \phi^*]} \left(1 + i \int d^4x j^\mu \partial_\mu \alpha \right). \tag{3.166}$$

Plugging this back into the propagator gives that the variation $\delta\phi(x)\phi^*(y)$ takes the form

$$\left\langle \phi(x) \phi^*(y) \left[\left(1 + i \int d^4x j^\mu \partial_\mu \alpha \right) \left[1 + i\alpha(x) - i\alpha(y)\right] - 1 \right] \right\rangle_{\text{full}} = 0. \tag{3.167}$$

Keeping only linear terms in $\alpha(x)$ gives

$$
\begin{aligned}
0 &= \int d^4 z \langle \phi(x)\phi^*(y) j^\mu(z) \partial_\mu \alpha(z)\rangle_{\text{full}} + \langle \phi(x)\phi(y)\alpha(x)\rangle_{\text{full}} - \langle \phi(x)\phi(y)\alpha(y)\rangle_{\text{full}} \\
&= \int d^4 z \alpha(z) \left[-\partial_\mu \langle \phi(x)\phi^*(y) j^\mu(z)\rangle_{\text{full}} + \langle \phi(x)\phi^*(y)\rangle_{\text{full}}\, \delta(x-z) - \langle \phi(x)\phi^*(y)\rangle_{\text{full}}\, \delta(y-z)\right].
\end{aligned} \tag{3.168}
$$

Once again, since α is an arbitrary function, the integrand must vanish along with the integral so we are left with

$$
\boxed{\partial_\mu \langle \phi(x)\phi^*(y) j^\mu(z)\rangle_{\text{full}} = \langle \phi(x)\phi^*(y)\rangle_{\text{full}}\, \delta(x-z) - \langle \phi(x)\phi^*(y)\rangle_{\text{full}}\, \delta(y-z)}. \tag{3.169}
$$

where ∂_μ only acts on the z coordinates. This is the Ward-Takahashi identity. Looking rather complex, the Ward-Takahashi identity simplifies greatly in momentum space by Fourier transforms (derivation not included here) to give

$$
\boxed{\Gamma_{3,\text{full}} = \tilde{G}_{\text{full}}(\boldsymbol{P}+\boldsymbol{K}) - \tilde{G}_{\text{full}}(\boldsymbol{P})}, \tag{3.170}
$$

where $\Gamma_{3,\text{full}}$ denotes the full 3-vertex of the theory, $G_{\text{full}}(\boldsymbol{x}-\boldsymbol{y}) = \langle \phi(x)\phi^*(y)\rangle_{\text{full}}$ and tildes denote Fourier transforms. So we see that for the complex scalar field, the Ward identity grants us a relation between the full propagator (RHS) and the full 3-vertex (LHS) which are exact relations in QFT.

§3.9 The O(N)-Vector Model

In the previous section, we have been dealing with complex scalar fields which are effectively a 2-component scalar fields consisting of components $\phi_1(x)$ and $\phi_2(x)$. In this section, we are going to generalize this notion to scalar fields with N-components. For a 2-component scalar field, recall that the $U(1)$ transformation was a symmetry of the action. This transformation can instead be written as a matrix-vector operation on the components of the scalar field as follows

$$
\phi'(x) = e^{i\alpha}\phi(x), \tag{3.171a}
$$

$$
\Rightarrow \quad \begin{bmatrix} \phi_1'(x) \\ \phi_2'(x) \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix}. \tag{3.171b}
$$

Written in this way, the transformation is now instead an element of $SO(2)$ or simply a rotation matrix in 2-dimensions. To generalize to an N-component scalar field, we then consider transformations which are elements of $SO(N)$, which then render the ϕ^4 theory Euclidean action (in 3+1 coordinate dimensions) as

$$
S_E = \int d^4 x_E \left[\frac{1}{2}\partial_a \boldsymbol{\phi} \cdot \partial_a \boldsymbol{\phi} + \frac{1}{2} m^2 \boldsymbol{\phi}\cdot\boldsymbol{\phi} + \frac{2\lambda}{N}(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2 \right], \tag{3.172}
$$

where bolded symbols indicate vectors (i.e. $\boldsymbol{\phi} = \vec{\phi}$). The QFT that arises from this generalization is known as the *$O(N)$-vector model*, with partition function given as

$$
Z = \int \mathcal{D}_{\boldsymbol{\phi}} e^{-S_E}, \tag{3.173}
$$

where of course in the cases where $N = 2$ and $N = 1$, we return to the complex and real scalar field QFTs respectively.

Note: A benefit of the $O(N)$-vector model is that the QFT can in fact be solved exactly in the limit of large N ($N \gg 1$). This is a rare case in which perturbation theory is **not** required to study the QFT. A useful reference for this topic can be found here.

To solve the $O(N)$ vector model in the large N limit, we insert the identity as a path integral over a δ-function into the partition function

$$\begin{aligned} Z &= \int \mathcal{D}_{\boldsymbol{\phi}} e^{-S_E} \times \mathbb{I} \\ &= \int \mathcal{D}_{\boldsymbol{\phi}} e^{-S_E} \int \mathcal{D}_\sigma \delta\left(\sigma - \boldsymbol{\phi}\cdot\boldsymbol{\phi}/N\right) \\ &= \int \mathcal{D}_{\boldsymbol{\phi}} e^{-S_E} \int \mathcal{D}_\sigma \int \mathcal{D}_\xi \exp\left(i \int d^4x\, \xi \cdot (\sigma - \boldsymbol{\phi}\cdot\boldsymbol{\phi}/N)\right) \\ &= \int \mathcal{D}_{\boldsymbol{\phi}} \mathcal{D}_\sigma \mathcal{D}_\xi \exp\left(-\frac{1}{2}\int d^4x \boldsymbol{\phi} \cdot \left[-\partial_a^2 + m^2 + \frac{2i\xi}{N}\right]\boldsymbol{\phi} - 2\lambda N \int d^4x \sigma^2 + i \int d^4x \xi\sigma\right) \\ &= \int \mathcal{D}_{\boldsymbol{\phi}} \mathcal{D}_\xi \exp\left(-\frac{1}{2}\int d^4x \boldsymbol{\phi} \cdot \left[-\partial_a^2 + m^2 + \frac{2i\xi}{N}\right]\boldsymbol{\phi} - \frac{1}{8\lambda N}\int d^4x \xi^2\right). \end{aligned} \tag{3.174}$$

If we then rescale the auxiliary field ξ we introduced to $\xi \to N\xi$, we get

$$Z = \int \mathcal{D}_{\boldsymbol{\phi}} \mathcal{D}_\xi \exp\left(-\frac{1}{2}\int d^4x \boldsymbol{\phi} \cdot \left[-\partial_a^2 + m^2 + 2i\xi\right]\boldsymbol{\phi} - \frac{N}{8\lambda}\int d^4x \xi^2\right). \tag{3.175}$$

Further decomposing the this auxiliary field into mean-field and fluctuation terms $\xi(x) = \bar{\xi} + \delta\xi(x)$, we get

$$Z = \int d\bar{\xi} \int \mathcal{D}_{\boldsymbol{\phi}} \mathcal{D}_{\delta\xi} \exp\left(-\frac{1}{2}\int d^4x \boldsymbol{\phi} \cdot \left[-\partial_a^2 + m^2 + 2i\xi\right]\boldsymbol{\phi} - \frac{N\beta V}{8\lambda}\bar{\xi}^2 - \frac{N}{8\lambda}\int d^4x \delta\xi^2\right). \tag{3.176}$$

So far, everything we have done has been exact for all N. It is now time to consider the large N limit where we note that the fluctuation term only contributes a order $\ln N$ term in the exponent (due to the path integral), whereas the mean-field term contributed a term of order N. As such, we are left with

$$\begin{aligned} \lim_{N \gg 1} Z &= \int d\bar{\xi} \int \mathcal{D}_{\boldsymbol{\phi}} \mathcal{D}_{\delta\xi} \exp\left(-\frac{1}{2}\int d^4x \boldsymbol{\phi} \cdot \left[-\partial_a^2 + m^2 + 2i\xi\right]\boldsymbol{\phi} - \frac{N\beta V}{8\lambda}\bar{\xi}^2\right) \\ &= \int d\bar{\xi} \exp\left(N \ln Z_{\text{free}}(T, \sqrt{m^2 + 2i\bar{\xi}}) - \frac{N\beta V}{8\lambda}\bar{\xi}^2\right), \end{aligned} \tag{3.177}$$

where the "mass" of the real scalar field is then $\sqrt{m^2 + 2i\bar{\xi}}$. The remaining integral can be evaluated with the saddle point approximation (which becomes exact when $N \to \infty$) which gives

$$\lim_{N\gg 1} Z = \exp\left(N\ln Z_{\text{free}}(T,\sqrt{m^2+2i\tilde{\xi}}) - \frac{N\beta V}{8\lambda}\tilde{\xi}^2\right), \tag{3.178}$$

where $\bar{\xi} = \tilde{\xi}$ is the saddle point. From the partition function, we can then derive the pressure (recalling $p = T\ln Z/V$) as

$$\boxed{p(T,m,\lambda) = N\left[p_{\text{free}}(T,\sqrt{m^2+2i\tilde{\xi}}) - \frac{\tilde{\xi}^2}{8\lambda}\right]}. \tag{3.179}$$

We stress again that this result is **exact** for the large N limit on the $O(N)$-vector model, where the result depends on the coupling λ, both explicitly and implicitly through the saddle point condition:

$$\frac{\partial}{\partial\tilde{\xi}}p_{\text{free}}\left(T,\sqrt{m^2+2i\tilde{\xi}}\right) - \frac{\tilde{\xi}}{4\lambda} = 0. \tag{3.180}$$

§3.9.1 Non-perturbative Renormalization

The exact result we have above is unfortunately, still divergent due to the divergence in the free pressure. For this, we will require a non-perturbative renormalization scheme. For a simplifcation of the technicalities of this procedure, we will set the mass parameter to $m = 0$. In the $\overline{\text{MS}}$ scheme, we obtained that the regularized pressure can be read off from Eq. (2.35), which reads

$$p_{\text{free}}(T,m) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\overline{\mu}^2 e^{3/2}}{m^2}\right)\right] - J_B(T,m). \tag{3.181}$$

Plugging in the new effective mass from the $O(N)$-vector model and setting $m = 0$ gives the saddle point condition

$$\frac{x}{8\pi^2}\left[\frac{1}{\varepsilon} + \frac{2\pi^2}{\lambda} + \ln\left(\frac{\overline{\mu}^2 e^{1/2}}{2x}\right)\right] - I_B(T,\sqrt{2x}) = 0, \tag{3.182}$$

where we defined $x \equiv i\tilde{\xi}$ for lighter notation. At this point, perturbative renormalization is not an option since the value of the saddle was determined non-perturbatively. In this case, it is possible to non-perturbatively renormalize the pressure by setting

$$\begin{aligned}
&\frac{2\pi^2}{\lambda_{\text{phys}}} = \frac{1}{\varepsilon} + \frac{2\pi^2}{\lambda}, \\
\Rightarrow\quad &\frac{x}{8\pi^2}\left[\frac{2\pi^2}{\lambda_{\text{phys}}} + \ln\left(\frac{\overline{\mu}^2\sqrt{e}}{2x}\right)\right] - I_B(T,\sqrt{2x}) = 0, \\
\Rightarrow\quad &p_{\text{renorm}}(T,\lambda) = N\left[\frac{x^2}{16\pi^2}\ln\left(\frac{\overline{\mu}^2 e^{\frac{3}{2}}}{2x}\right) - J_B(T,\sqrt{2x}) + \frac{x^2}{8\lambda_{\text{phys}}}\right].
\end{aligned} \tag{3.183}$$

So we find that in the large N limit, the $O(N)$-vector model is renormalizable, for which in the $m = 0$ (dimensional regularization) case, the theory only requires coupling constant renormalization and in particular, **no** cosmological constant counter term.

Let us now consider properties of this solution. Because a physical observable cannot be scale dependent, we must enforce the condition we did in Eq. (3.95), which for pressure reads

$$\overline{\mu}\frac{dp(T,\lambda_{\text{phys}})}{d\overline{\mu}} = Nx^2\left[\frac{1}{8\pi^2} + \frac{d}{d\ln\overline{\mu}}\frac{1}{8\lambda_{\text{phys}}}\right] = 0. \tag{3.184}$$

Once again using the definition of β is Sec. (3.7), we have that

$$\beta = \bar{\mu}\frac{\partial\lambda_{\text{phys}}(\bar{\mu})}{\partial\bar{\mu}} = \frac{\lambda^2_{\text{phys}}}{\pi^2}, \tag{3.185}$$

$$\Rightarrow \boxed{\lambda_{\text{phys}}(\bar{\mu}) = \frac{2\pi^2}{\ln\left(\mu_0^2/\bar{\mu}^2\right)}}, \tag{3.186}$$

where the two expressions above are exact in the large N limit. Here, μ_0 is the Landau pole of the theory which we defined earlier (i.e. the value of $\overline{\mu}$ such that $\lambda_{\text{phys}}(\bar{\mu}) \to \infty$).

Note: The $O(N)$-vector model does not have a good high energy ($\overline{\mu} \to \infty$) continuum limit since the theory becomes infinitely coupled at $\overline{\mu} = \mu_0$. So the $O(N)$-vector model is an "effective theory" valid for $\overline{\mu} \ll \mu_0$.

Unlike other theories, the $O(N)$-vector model has a finite *cosmological constant*, which is defined as the energy density of free space (vacuum pressure). This is computed as the zero temperature ($T = 0$) pressure

$$\begin{aligned} p_{\text{renorm}}(T=0,\lambda) &= \frac{Nx^2}{16\pi^2}\left[\ln\left(\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{2x}\right) + \frac{2\pi^2}{\lambda_{\text{phys}}}\right] \\ &= \frac{Nx^2}{16\pi^2}\ln\left(\frac{\mu_0^2 e^{\frac{3}{2}}}{2x}\right). \end{aligned} \tag{3.187}$$

where we plugged in the explicitly form of λ_{phys}. The saddle point solution is then

$$\begin{aligned} &\frac{x^2}{8\pi^2}\ln\left(\frac{\mu_0^2 e^{\frac{1}{2}}}{2x}\right) = 0, \\ \Rightarrow\quad & x = 0 \quad \text{or} \quad x = \frac{\mu_0^2 e^{\frac{1}{2}}}{2}, \\ \Rightarrow\quad & \boxed{p_{\text{renorm}}(T=0,\lambda) = 0}, \end{aligned} \tag{3.188}$$

where we discarded the $x \neq 0$ solution since it is proportional to the Landau pole which is where the theory fails (is unreliable). This result, albeit appealing may be misleading since we recall that in dimensional regularization, only logarithmic divergences are registered. By contrast, cut-off regularization would contain cut-off terms such proportional to Λ^2, Λ^4, etc., which would require additional counter term. As such, **the result of the cosmological constant obtained from dimensional regularization need to be interpreted with great care!**

§3.9.2 Finite Temperature Pressure

Let's go back to looking at the pressure at finite temperatures. To do so, we will first adopt the notation where we write the temperature as

$$T = \mu_0 e^{-\chi}, \tag{3.189}$$

which indicates that the temperature is exponentially suppressed from μ_0 by virtue of $\chi \gg 1$. With this, we have that the pressure and saddle point conditions can be written as

$$p_{\text{renorm}}(T) = N\left[\frac{x^2}{16\pi^2}\ln\left(\frac{T^2 e^{\frac{3}{2}+2\chi}}{2x}\right) - J_B(T, \sqrt{2x})\right], \tag{3.190a}$$

$$\frac{x}{8\pi^2}\ln\left(\frac{T^2 e^{\frac{1}{2}+2\chi}}{2x}\right) - I_B(T, \sqrt{2x}) = 0. \tag{3.190b}$$

We also note that the functions J_B and I_B can be written as sums over modified Bessel functions

$$J_B(T, m) = -\frac{m^2T^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2(nm\beta), \tag{3.191a}$$

$$I_B(T, m) = \frac{mT}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}K_1(nm\beta). \tag{3.191b}$$

Since x has to be proportional to T^2 have mass-dimension 2, we can write $x = m_B^2(\chi)T^2$, where $m_B(\chi)$ is some generalized coupling. The saddle point condition on x now becomes a condition on $m_B(\chi)$, which can be solved numerically. Plugging this back into the pressure gives the plot shown in Fig. 3.11.

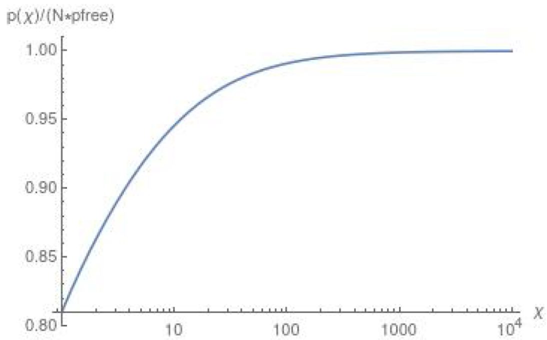


Figure 3.11: Plot of the renormalized finite temperature pressure (normalized by the free pressure) against the temperature parameter χ.

Chapter 4

Fermions

*Thus far, we have dealt with scalar field theories which are commonly referred to as spin-0 fields (a.k.a. bosonic fields). However, most of the fundamental fields that arise in nature are **not** scalar fields (with the exception of the Higgs field). As such, wed want to understand how to set-up systems with fields that have non-zero spin. In this chapter, we will be moving into a study of fermionic quantum field theories by considering the path integral for fermionic (spin-$\frac{1}{2}$) quantum fields.*

§4.1 From Bosons to Fermions

As a first step, it would be useful to review some of the steps taken to construct bosonic quantum field theories. For bosons, we started by taking the harmonic oscillator Hamiltonian

$$\begin{aligned}\hat{H} &= -\frac{1}{2m}\partial_x^2 + \frac{1}{2}m\omega^2\hat{x}^2 \\ &= \frac{\omega}{2}\left(-\partial_q^2 + \hat{q}^2\right) \\ &= \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),\end{aligned} \tag{4.1}$$

where $\hat{q} = \hat{x}\sqrt{m\omega}$ and

$$\hat{a} = \frac{\partial_q + \hat{q}}{\sqrt{2}}, \tag{4.2a}$$

$$\hat{a}^\dagger = \frac{-\partial_q + \hat{q}}{\sqrt{2}}, \tag{4.2b}$$

known as the ladder operators. These ladder operators have specific commutation relations which are written as

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \tag{4.3a}$$

$$\left[\hat{a}, \hat{a}\right] = \left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0, \tag{4.3b}$$

which allow us to write the Hamiltonian as

$$\hat{H} = \frac{\omega}{2}\left\{\hat{a}^\dagger, \hat{a}\right\}, \tag{4.4}$$

where $\{\ldots,\ldots\}$ denotes the anti-commutator. The ladder operators can of course raise and lower the state of the system via the relations

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \tag{4.5a}$$

$$\hat{a}^\dagger = \sqrt{n+1}\left|n+1\right\rangle. \tag{4.5b}$$

The partition function for this system is computed as

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} \left\langle n\right| e^{-\beta\hat{H}} \left|n\right\rangle \\ &= \sum_{n=0}^{\infty} e^{-\beta\omega\left(n+\frac{1}{2}\right)} \\ &= \frac{1}{\sinh(\beta\omega/2)}. \end{aligned} \tag{4.6}$$

Fermions on the other hand, are defined by different commutation relations from those of bosons. They instead follow the analogous anti-commutation relations

$$\left\{\hat{a}_f, \hat{a}_f^\dagger\right\} = 1, \tag{4.7a}$$

$$\left\{\hat{a}_f, \hat{a}_f\right\} = \left\{\hat{a}_f^\dagger, \hat{a}_f^\dagger\right\} = 0, \tag{4.7b}$$

which will allow us to write the fermionic Hamiltonian as

$$\hat{H}_f = \frac{\omega}{2}\left[\hat{a}_f^\dagger, \hat{a}_f\right] = \omega\left(\hat{a}_f^\dagger\hat{a}_f - \frac{1}{2}\right). \tag{4.8}$$

The anti-commutator relations for fermions imply that there are only **two** energy eigenstates in the Hilbert space since

$$\begin{aligned} \hat{a}_f^\dagger\left|1\right\rangle &= \hat{a}_f^\dagger\hat{a}_f^\dagger\left|0\right\rangle \\ &= \frac{1}{2}\left\{\hat{a}_f^\dagger, \hat{a}_f^\dagger\right\}\left|0\right\rangle = 0. \end{aligned} \tag{4.9}$$

So only the states $|0\rangle$ and $|1\rangle$ are permissible. The fermionic partition function is then computed as a sum over just these two states

$$\begin{aligned} Z_f &= \left\langle 0\right| e^{\frac{\beta\omega}{2}} \left|0\right\rangle + \left\langle 1\right| e^{\frac{\beta\omega}{2} - \beta\omega\hat{a}_f^\dagger\hat{a}_f} \left|1\right\rangle \\ &= e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}} \\ &= 2\cosh\left(\frac{\beta\omega}{2}\right). \end{aligned} \tag{4.10}$$

For fermions, we can also define specific bra and ket states built from the fermionic ladder operators through the use of Grassmann variables (see App. A) c and c^*:

$$|c\rangle \equiv e^{-c\hat{a}^\dagger} |0\rangle = \left(1 - c\hat{a}^\dagger\right) |0\rangle , \tag{4.11a}$$

$$\langle c| \equiv \langle 0| e^{-c^*\hat{a}} = \langle 0| \left(1 - c^*\hat{a}\right) , \tag{4.11b}$$

such that

$$\hat{a} |c\rangle = c |0\rangle , \tag{4.12a}$$

$$\langle c| \hat{a}^\dagger = \langle 0| c^* . \tag{4.12b}$$

Such states have transition amplitudes

$$\begin{aligned} \langle c'|c\rangle &= \langle 0| \left(1 - c^*\hat{a}\right) \left(1 - c\hat{a}^\dagger\right) |0\rangle \\ &= 1 + \langle 0| \hat{a}c'^* c\hat{a}^\dagger |0\rangle \\ &= 1 + \langle 0| c'^* c |0\rangle \\ &= 1 + c'^* c = e^{c'^* c}, \end{aligned} \tag{4.13}$$

where we demanded that the fermionic ladder operators also anti-commute with the grassmann variables. With these definitions and using the properties of Grassmann variables, we have the identity

$$\boxed{\int dc^* dc \, e^{-c^* c} |c\rangle \langle c| = \mathbb{I}}, \tag{4.14}$$

implying that this relation above generalizes the completeness relation for commuting systems to anti-commuting system. Another useful identity involving bosonic operators and Grassmann integrals is

$$\boxed{\int dc^* dc \, e^{-c^* c} \langle -c| \hat{A} |c\rangle = \mathrm{Tr}\left\{\hat{A}\right\}}, \tag{4.15}$$

which defines the trace of a bosonic operator in a fermionic system. With this, we have all the necessary tools to construct the path integral for fermions.

§4.1.1 Fermionic Path Integrals

We start with the partition function for fermions written in terms of the trace as

$$Z_f = \mathrm{Tr}\left\{e^{-\beta\hat{H}}\right\} = \int dc^* dc \, e^{-c^* c} \langle -c| e^{-\beta\hat{H}} |c\rangle . \tag{4.16}$$

Next, we follow the scheme in Sec. 1.3 and split the Boltzmann factor into a product of $N \gg 1$ pieces $\epsilon \equiv \beta/N$, such that

$$e^{-\beta\hat{H}} = e^{-\epsilon\hat{H}} e^{-\epsilon\hat{H}} e^{-\epsilon\hat{H}} \ldots e^{-\epsilon\hat{H}}, \tag{4.17}$$

for which we then insert the fermionic completeness relation of Eq. (4.14), which gives objects

$$\begin{aligned} e^{-c_j^* c_j} \langle c_j | \, e^{-\epsilon \hat{H}} \, | c_{j-1} \rangle &= e^{-c_j^* c_j} e^{-\epsilon H(c_j^*, c_{j-1})} \langle c_j | c_{j-1} \rangle \\ &= \exp\left[-\epsilon \left(\frac{c_j^*(c_j - c_{j-1})}{\epsilon} + H(c_j^*, c_{j-1}) \right) \right], \end{aligned} \tag{4.18}$$

where $H(c_j^*, c_{j-1})$ is no longer an operator but a function evaluated at the associated values of c_j^* and c_j. The resulting partition function can be written as

$$Z_f = \int dc_N^* dc_N \int dc_{N-1}^* dc_{N-1} \ldots \int dc_1^* dc_1 e^{-S_E}, \tag{4.19a}$$

$$\text{where} \quad S_E = \epsilon \sum_{j=1}^{N} \left[c_{j+1}^* \frac{c_{j+1} - c_j}{\epsilon} + H(c_{j+1}^*, c_j) \right]. \tag{4.19b}$$

For fermions, we then enforce *anti-periodic boundary conditions* which mean that we identify $c_{N+1} = -c_1$ and $c_{N+1}^* = -c_1^*$. As before, we can think of this discretization of β with N points as points on a thermal circle, which in the limit $N \to \infty$ renders these c Grassmann variables a function of the imaginary time parameter $\tau \in [0, \beta]$, such that $c(\beta) = -c(0)$ and $c^*(\beta) = -c^*(0)$. So the partition function in this continuum limit becomes

$$\boxed{Z_f = \int \mathcal{D}_{c^*} \mathcal{D}_c \, \exp\left[\int_0^\beta d\tau \left(c^* \frac{dc}{d\tau} + H(c^*, c) \right) \right]}. \tag{4.20}$$

§4.1.2 The Dirac Equation

Recall that in Sec. 2.3.1, we introduced the equation of motion for free scalar fields known as the Klein-Gordon equation

$$\left(\Box - m^2 \right) \phi = 0, \tag{4.21}$$

which was derived from the Euler-Lagrange equation given in Eq. (2.8). In this section, we want to once again derive an equation of motion (wave-equation) but this time, for fermionic fields. For non-relativistic particles, the wave equation for fermions is simply the Schrödinger's equation, written for free particles as

$$i\partial_t \psi = -\frac{1}{2m} \boldsymbol{\nabla}^2 \psi. \tag{4.22}$$

Unfortunately, the time and space derivatives in this equation appear asymmetrically and hence is clearly **not** Lorentz invariant (unable to be used for a relativistic QFT). To try and fix this, we note that the usual right hand side of the Schrödinger equation is the Hamiltonian acting on ψ, so we attempt to replace it with the relativistic free-particle energy

$$E = \sqrt{m^2 + p^2} \tag{4.23}$$

$$\Rightarrow \quad i\partial_t \psi = \sqrt{m^2 + \hat{p}^2} \psi. \tag{4.24}$$

This equation still has some issues as it is non-local and still not Lorentz invariant. If however, we apply the operators on both sides twice, we get the expression

$$\left(-\partial_t^2 - \hat{p}^2 - m^2\right)\psi = \left(\Box - m^2\right) = 0, \tag{4.25}$$

which is once again the Klein-Gordon equation! But what about fermions? Dirac proposed that perhaps the wrong square root of the Klein-Gordon equation was taken in the earlier step, and perhaps we just have to take the “correct” square root (to get an equation that was linear in derivatives). To do so, he used the ansatz

$$i\partial_t\psi = \hat{H}\psi = \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m\right)\psi, \tag{4.26}$$

where $\boldsymbol{\alpha}$ and β are constants of the ansatz. In order to ensure that this ansatz was consistent, applying the operators twice as before must yield the Klein-Gordon equation, so we try

$$\begin{aligned} -\partial_t^2\psi &= \left(-i\alpha_j\partial_j + \beta m\right)\left(-i\alpha_k\partial_k + \beta m\right)\psi \\ &= \left[\alpha_j\alpha_k\partial_j\partial_k - i\left(\alpha_j\beta + \beta\alpha_j\right)\partial_j + \beta^2 m^2\right]\psi. \end{aligned} \tag{4.27}$$

Clearly for ordinary numbers α_j and β, the cross term will not vanish and we will not retrieve the Klein-Gordon equation. But Dirac realized that consistency could be acheived if these were in fact not numbers, but matrices which satisfied

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}\mathbb{I}, \tag{4.28a}$$

$$\{\alpha_j, \beta\} = 0, \tag{4.28b}$$

$$\beta^2 = \mathbb{I}. \tag{4.28c}$$

In modern notation, we can define what are known as the *Dirac matrices* via $\gamma^\nu = [\beta, \beta\boldsymbol{\alpha}]$. Dirac matrices fullfill the following anti-commutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}\mathbb{I}, \tag{4.29}$$

which constitutes what is known as a *Clifford algebra* ($g^{\mu\nu}$ being the metric tensor). We note that the timelike γ-matrices are Hermitian, while the spacelike γ-matrices are anti-Hermitian:

$$(\gamma^0)^\dagger = \gamma^0, \tag{4.30a}$$

$$(\gamma^j)^\dagger = \gamma^j, \tag{4.30b}$$

$$\Rightarrow \quad \boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}. \tag{4.30c}$$

With the Dirac matrices, the Dirac equation from taking the “correct” square root is

$$\boxed{\left(i\partial_\mu\gamma^\mu - m\right)\psi = 0}, \tag{4.31}$$

which fulfills the property

$$\begin{aligned} \left(i\partial_\mu\gamma^\mu + m\right)\left(i\partial_\mu\gamma^\mu - m\right)\psi &= \left(-\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu - m^2\right)\psi \\ &= \left(-\frac{1}{2}\partial_\mu\partial_\nu\left\{\gamma^\mu, \gamma^\nu\right\} - m^2\right)\psi \\ &= \left(\partial_\mu\partial_\nu g^{\mu\nu}\mathbb{I} - m^2\right)\psi \\ &= \left(\Box - m^2\right)\psi. \end{aligned} \tag{4.32}$$

Note that although this appears as the Klein-Gordon equation, the object in question here ψ must have multiple components unlike ϕ. We thus call ψ the *spinor field*, which is a relativistic extension of the quantum wave function. There are in fact different ways to construct γ^μ such that it satisfies the Clifford algebra, but the matrix representation of these in $3+1$ dimensions is given by

$$\boxed{\gamma^0 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & \hat{\sigma}_j \\ -\hat{\sigma}_j & 0 \end{bmatrix}}, \tag{4.33}$$

where $\hat{\sigma}_j$ are the Pauli matrices. Since each component of γ^μ is a 4×4 matrix, the spinor field will be represented as a 4-component vector which we often denote with index α:

$$\psi_\alpha = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}. \tag{4.34}$$

Also for lighter notation, since the combination $\partial_\mu \gamma^\mu$ appears so often, we often denote this as

$$\partial_\mu \gamma^\mu \equiv \not{\partial}, \tag{4.35}$$

read as the Dirac-slash. For complex conjugation, we define the *Dirac adjoint* of ψ as

$$\overline{\psi} \equiv \psi^\dagger \gamma^0. \tag{4.36}$$

Strangely enough, the route we have taken to deriving the Dirac equation has led us to attaining an equation of motion of ψ without knowing the Lagrangian density. As such, we can reverse engineer the equation, treating ψ and $\overline{\psi}$ as independent variables which each satisfy the Euler-Lagrange equations such that

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0, \tag{4.37}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = 0, \tag{4.38}$$

$$\Rightarrow \quad \mathcal{L} = \overline{\psi}\left(i\not{\partial} - m\right)\psi. \tag{4.39}$$

From this, we can also construct the Hamiltonian density as

$$\begin{aligned} \mathcal{H} &= \Pi \partial_0 \psi - \mathcal{L}, \\ &= i\overline{\psi}^\dagger \partial_0 \psi - \overline{\psi}\left(i\not{\partial} - m\right)\psi \\ &= \overline{\psi}\left(-i\gamma^j \partial_j + m\right)\psi. \end{aligned} \tag{4.40}$$

where $\Pi = \partial\mathcal{L}/\partial(\partial_0\psi) = \overline{\psi} i \gamma^0 = i\overline{\psi}^\dagger$ is the conjugate momentum. Integrating this over all coordinates gives the Hamiltonian

$$H = \int d^3x \mathcal{H} = \int d^3x \, \overline{\psi}\left(-i\gamma^j \partial_j + m\right)\psi. \tag{4.41}$$

For all intends and purposes, ψ and $\psi^\dagger$ here are taken as classical (but anti-commuting) fields. Only when we quantize the theory do these become operators which play the role of the fermionic ladder operators (as in canonical second quantization). To then get an expression for the path integral, we identify $\psi, \psi^\dagger$ with c, c^* to get

$$Z_f = \int \mathcal{D}_{\psi^\dagger}\mathcal{D}_\psi \, e^{-S_E}, \tag{4.42a}$$

$$\text{where} \quad S_E = \int_0^\beta \int d^3x \, \psi^\dagger \left[\partial_\tau - i\gamma^0\gamma^j\partial_j + \gamma^0 m\right]\psi, \tag{4.42b}$$

with S_E known as the Euclidean action for Dirac fermions.

Note: To reiterate the anti-periodic boundary conditions, we once again state the identification on the thermal circle $\psi(0, \boldsymbol{x}) = \psi(\beta, \boldsymbol{x})$.

In the Euclidean picture (converting to imaginary time via a Wick rotation), we can define the Euclidean Dirac matrices as

$$\gamma_0^E = \gamma_0, \tag{4.43a}$$

$$\gamma_j^E = -i\gamma^j, \tag{4.43b}$$

which obey the anti-commutation and Hermiticity relations

$$\left\{\gamma_a^E, \gamma_b^E\right\} = 2\delta_{ab}\mathbb{I}, \tag{4.44a}$$

$$(\gamma_a^E)^\dagger = \gamma_a^E. \tag{4.44b}$$

Finally, we can write the partition function in terms of the Dirac adjoint as

$$Z_f \propto \int \mathcal{D}_{\overline{\psi}}\mathcal{D}_\psi \, \exp\left[-\int d^4x_E \, \overline{\psi}\left(\gamma_a^E\partial_a + m\right)\psi\right], \tag{4.45}$$

where proportionality is due to the fact that we did not include a constant matrix out front which arises from the Jacobian. Now, we look to evaluating this path integral. To do so, we first write the fermionic field as a Fourier series

$$\psi(\tau, \boldsymbol{x}) = \frac{T}{V}\sum_n \sum_{\boldsymbol{k}} e^{i\tilde{\omega}_n\tau + i\boldsymbol{K}\cdot x}\tilde{\psi}(\omega_n, \boldsymbol{k}), \tag{4.46}$$

where V is the volume of space and the *fermionic Matsubara frequencies* are given as

$$\tilde{\omega}_n = (2n+1)\pi T, \tag{4.47}$$

due to anti-periodicity. We can also define a fermionic Euclidean momentum by taking the boundary conditions in space as periodic (although those in time are anti-periodic) as

$$\tilde{\boldsymbol{P}}_a \equiv (\omega_n, \boldsymbol{p}). \tag{4.48}$$

With this, we can write the Fourier expansion as

$$\psi(\boldsymbol{x}) = \frac{1}{\beta V} \sum_{\tilde{\boldsymbol{K}}} e^{i\tilde{\boldsymbol{K}}\cdot\boldsymbol{x}} \tilde{\psi}(\tilde{\boldsymbol{K}}). \tag{4.49}$$

The Dirac adjoint of this is then

$$\overline{\psi}(\boldsymbol{x}) = \frac{1}{\beta V} \sum_{\tilde{\boldsymbol{K}}} e^{-i\tilde{\boldsymbol{K}}\cdot\boldsymbol{x}} \tilde{\overline{\psi}}(\tilde{\boldsymbol{K}}), \tag{4.50}$$

which gives the Euclidean action

$$\begin{aligned} S_E &= \frac{1}{\beta^2 V^2} \int d^4x \sum_{\tilde{\boldsymbol{K}},\tilde{\boldsymbol{P}}} e^{i\boldsymbol{x}\cdot(\tilde{\boldsymbol{P}}-\tilde{\boldsymbol{K}})} \tilde{\overline{\psi}}(\tilde{\boldsymbol{K}}) \left[i\gamma_a^E \tilde{P}_a + m \right] \tilde{\psi}(\tilde{\boldsymbol{P}}) \\ &= \frac{1}{\beta V} \sum_{\tilde{\boldsymbol{P}}} \tilde{\overline{\psi}}(\tilde{\boldsymbol{P}}) \left[i\gamma_a^E \tilde{P}_a + m \right] \tilde{\psi}(\tilde{\boldsymbol{P}}) \\ &= \frac{1}{\beta V} \sum_{\tilde{\boldsymbol{P}}} \tilde{\overline{\psi}}(\tilde{\boldsymbol{P}}) \left[i\tilde{\not{\boldsymbol{P}}} + m \right] \tilde{\psi}(\tilde{\boldsymbol{P}}). \end{aligned} \tag{4.51}$$

Putting this back into the partition function and dropping all the tildes on the Fourier transformed field terms, we are left with

$$Z_f \propto \int \mathcal{D}_{\overline{\psi}} \mathcal{D}_{\psi} \, \exp\left[-\frac{1}{\beta V} \sum_{\tilde{\boldsymbol{P}}} \tilde{\psi}(\tilde{\boldsymbol{P}}) \left(i\tilde{\not{\boldsymbol{P}}} + m \right) \psi(\tilde{\boldsymbol{P}}) \right], \tag{4.52}$$

which is a Gaussian-type integral and can be evaluated using the multidimensional Grassmann integration rules for Gaussians in App. A to give

$$Z_f = \tilde{C} \prod_{\tilde{\boldsymbol{P}}} \det \left[i\tilde{\not{\boldsymbol{P}}} + m \right], \tag{4.53}$$

where the determinant is over spinor space (spanned by the Dirac matrices γ_a^E) and $\tilde{C}$ is an overall constant. Since Z_f is real, we have that

$$\begin{aligned} \det \left[i\tilde{\not{\boldsymbol{P}}} + m \right] &= \det \sqrt{ \left[-i\tilde{\not{\boldsymbol{P}}} + m \right] \left[i\tilde{\not{\boldsymbol{P}}} + m \right] } \\ &= \det \sqrt{ \tilde{\not{\boldsymbol{P}}}^2 + m^2 } \\ &= \det \sqrt{ \left(\tilde{P}^2 + m^2 \right) \mathbb{I}_4 } \\ &= \left(\tilde{P}^2 + m^2 \right)^2. \end{aligned} \tag{4.54}$$

So finally, we have

$$
\begin{aligned}
Z_f &= \tilde{C} \prod_{\tilde{\boldsymbol{P}}} \left(\tilde{P}^2 + m^2 \right)^2 \\
&= \tilde{C} \prod_{n=-\infty}^{\infty} \prod_{\boldsymbol{p}} \left(\tilde{\omega}_n^2 + E_p^2 \right)^2 , \\
\Rightarrow \quad & \boxed{Z_f = \tilde{C} \prod_{\boldsymbol{p}} \left[2 \cosh \left(\frac{\beta E_p}{2} \right) \right]^4 = \exp \left(4 \sum_{\boldsymbol{p}} \left[\frac{\beta E_p}{2} + \ln \left(1 + e^{-\beta E_p} \right) \right] \right)} ,
\end{aligned}
\tag{4.55}
$$

where $E_p = \sqrt{p^2 + m^2}$. With this, we can derive the fermionic free pressure

$$
p^f_{\text{free}} = \frac{4}{V} \sum_{\boldsymbol{p}} \left[\frac{E_p}{2} + T \ln \left(1 + e^{-\beta E_p} \right) \right] , \tag{4.56}
$$

which in the large volume limit, becomes

$$
\boxed{p^f_{\text{free}} = 4 \int \frac{d^3 p}{(2\pi)^3} \left[\frac{E_p}{2} + T \ln \left(1 + e^{-\beta E_p} \right) \right]} . \tag{4.57}
$$

Notice the resemblence of these result with that for bosons in Eqs. (2.19) and (2.21). Most noticeable are the factor 4 and change in sign of the exponent in the logarithm of the integrand. Because of this, the bosonic and fermionic pressures are related in the zero-temperature limit ($T = 0$) by

$$
p^f_{\text{free}}(T = 0) = -4 p^b_{\text{free}}(T = 0). \tag{4.58}
$$

Therefore, one fermionic degree of freedom (see Sec. 3.8) is one-quarter the free fermionic pressure and the negative of a bosonic degree of freedom. Hence, a theory of an equal number of bosons and fermions is said to have vanishing pressure (cosmological constant).

Chapter 5

Gauge Fields

Thus far, we have dealt with quantum field theories for scalar fields and fermions. To extend our discuss of QFT, we will need to now address an important class of fields known as gauge fields. Gauge fields are in fact found in nature and give rise to important theories such as quantum electrodynamics.

§5.1 Introduction

To begin, we recall that the Euclidean action for a complex field was given as

$$S_E = \int d^4 x_E \left[\partial_a \phi \partial_a \phi^* + V(\sqrt{\phi \phi^*}) \right], \tag{5.1}$$

for which it was Lorentz and $U(1)$ invariant. In the $U(1)$ symmetries we considered, the transformation $\phi(x) \to e^{i\alpha}\phi(x)$ was not x dependent (α is x independent), so we refer this as a *global $U(1)$ transformation* or a *gauge transformation* (because we are "re-gauging" what it means to have a certain value of ϕ). If α was instead dependent on x (i.e. $\phi(x) \to e^{i\alpha(x)}\phi(x)$), this transformation would be known as a *local gauge transformation.* The action for the complex scalar field above is however **not** local gauge invariant, so it would be prudent for us to look for an action that is. First, we consider the result of a local gauge transformation on $\partial_a \phi$:

$$\partial_a \phi \quad \to \quad \partial_a \left(e^{i\alpha(x)} \phi \right) = e^{i\alpha(x)} \left[\phi \partial_a i\alpha(x) + \partial_a \phi \right]. \tag{5.2}$$

We call this the *local $U(1)$ gauge field.* We see that in this case, there is an extra term of $\phi \partial_a i\alpha(x)$ which arises unlike in global $U(1)$ transformations. The question now is then, how do we modify the action such that these terms disappear? One such possibility is adding a **new field** A_a, appropriately to the action:

$$S_E = \int d^4 x \left[\left(\partial_a + iA_a \right) \phi \left(\partial_a - iA_a \right) \phi^* + V(\sqrt{\phi \phi^*}) \right], \tag{5.3}$$

in which this new field must transform under local gauge transformations as

$$A_a(x) \quad \to \quad A_a(x) - \partial_a \alpha(x). \tag{5.4}$$

We now introduce a new operator associated to this new field:

$$D_a \equiv \partial_a - iA_a, \tag{5.5}$$

which we call the *gauge-covariant derivative*. This shortens the notation of the action to

$$S_E = \int d^4x \left[D_a\phi \left(D_a\phi\right)^* + V(\sqrt{\phi\phi^*}) \right]. \tag{5.6}$$

This action is now manifestly real, Lorentz invariant and local gauge invariant. Extending this idea, we can now ask what other terms could be allowed in the action associated to $A_a(x)$? For one, we can consider an anti-symmetric tensor

$$F_{ab} = \partial_a A_b - \partial_b A_a, \tag{5.7}$$

which is local gauge and Lorentz invariant. We term this tensor as the *field strength tensor*, which is conventionally added to the action with a factor 1/4 to give

$$S_E = \int d^4x_E \left[D_a\phi \left(D_a\phi\right)^* + V(\sqrt{\phi\phi^*}) + \frac{1}{4} F_{ab}F_{ab} \right]. \tag{5.8}$$

Alternatively, we could also consider the addition of this tensor along with the 4-dimensional Levi-Civita tensor, $F_{ab}\epsilon_{abcd}F_{cd}$. This however would violate an additional symmetry known as *parity*, which we will cater to a later time for further discussion. For now, we shall simply disallow the addition of this term to our action. The action in Eq. (5.8) constitutes a theory known as *scalar electrodynamics*. This is so because it turns out that the resulting equations of motion for the gauge field $A_a(x)$ work out to be exactly Maxwell's equations for classical electrodynamics. The gauge field is coupled to the complex scalar field via the gauge-covariant coupling, which renders the complex scalar field as the "*matter content*" in the theory (i.e. the equivalent of electrons in electromagnetism).

> **Note**: Since electrons are fermions, this scalar-gauge theory is not "real", and simply a toy model which is simpler to analyze.

To now promote this theory to a QFT, we will have to plug the action into a path integral

$$Z = \int \mathcal{D}_{\phi^*}\mathcal{D}_{\phi}\mathcal{D}_A e^{-S_E}. \tag{5.9}$$

However, difficulties arise that were not present for scalar fields and fermions when we try to use the same methods for gauge fields.

§5.2 Quantum Electrodynamics

Quantum electrodynamics (QED) is a relativistic quantum theory of light and matter interaction (i.e. the quantum mechanical extension of classical electrodynamics). Since the scalar field is in fact not completely representative of the actual theory of QED (where fermionic fields are necessary), we will just concentrate on the gauge field term so that

$$S_E = \frac{1}{4}\int d^4x_E F_{ab}F_{ab}, \tag{5.10}$$

where as defined earlier, $F_{ab} = \partial_a A_b - \partial_b A_a$.

Note: Including the actual matter associated spinor field into the theory does not change at all the procedure to which the path integral for the gauge field is solved.

It is only natural then that the extension of this theory to a QFT would be to write

$$Z = \int \mathcal{D}_A e^{-S_E}, \tag{5.11}$$

for which since the gauge field term is quadratic, we consider its Fourier transform

$$A_a(x) = \frac{1}{\beta V}\sum_{\boldsymbol{K}} e^{i\boldsymbol{K}\cdot x}\tilde{A}_a(x). \tag{5.12}$$

The Euclidean action then becomes

$$S_E = \frac{1}{2\beta V}\sum_{\boldsymbol{K}} \tilde{A}_a(\boldsymbol{K})\left[K^2\delta_{a,b} - K_aK_b\right]\tilde{A}_b(-\boldsymbol{K}). \tag{5.13}$$

Neglecting the Jacobian from the Fourier transform (since it is just a constant) leaves us with

$$Z = \prod_{\boldsymbol{K}} \det\left\{K^2\delta_{a,b} - K_aK_b\right\}^{-\frac{1}{2}}. \tag{5.14}$$

This result seems fine except for the fact that the matrix $K^2\delta_{a,b} - K_aK_b$, has a vanishing eigenvalue associated to eigenvector K_a:

$$\left[K^2\delta_{a,b} - K_aK_b\right]K_a = 0. \tag{5.15}$$

This means that the matrix is **not** invertible and $\det\left\{K^2\delta_{a,b} - K_aK_b\right\} = 0$, which leads to a divergence in Z. The reason why this happens, is in fact because of the invariance of the action under the local gauge transformation, implying essentially that there is an infinite class of gauge fields which the integral goes over, i.e.:

$$Z \sim \int_{-\infty}^{\infty} d\alpha\, e^{\text{constant}} \to \infty, \tag{5.16}$$

so Z indeed converges. There are of course ways to better understand this theory such as regularization of the integral by inserting cutoffs, which in this case is referred to *compactifying* the range of the gauge parameter leading to a *compact* $U(1)$ *gauge theory*. Such methods result in interesting properties such as self-interacting photons, which unfortunately does not occur in nature and so is not a very accurate means of dedaling with this divergence.

§5.2.1 The Fadeev-Popov Trick

Since the partition function diverges as a result of the gauge invariance, there is an easy (albeit not very elegant) way to break this gauge symmetry which fixes a gauge. Examples of these gauge conditions are:

$$\text{Coulomb gauge}: \quad \partial_i A_i = 0, \tag{5.17a}$$

$$\text{Landau gauge}: \quad \partial_a A_a = 0. \tag{5.17b}$$

In general, we can denote an arbitrary gauge condition as $G[A] = f$. By adding such conditions, the path integral would not only run over **inequivalent** gauge fields (gf) which we denote $\overline{A}$:

$$Z_{\text{gf}} = \int \mathcal{D}_{\overline{A}} e^{-S_E[\overline{A}]}. \tag{5.18}$$

From here, we insert the gauge condition into the expression by writing

$$\begin{aligned} Z_{\text{gf}} &= \int \mathcal{D}_{\overline{A}} \mathcal{D}_G \delta(G[A] - f) e^{-S_E[\overline{A}]} \\ &= \int \mathcal{D}_{\overline{A}} \mathcal{D}_\alpha \mathcal{D}_f \delta(G[A] - f) \det\left\{\frac{\partial G[A]}{\partial \alpha}\right\} e^{-S_E[\overline{A}]} \\ &= \int \mathcal{D}_A \mathcal{D}_f \delta(G[A] - f) \det\left\{\frac{\partial G[A]}{\partial \alpha}\right\} e^{-\frac{1}{2\xi}\int d^4x_E f^2(x)} e^{-S_E[A]} \\ &= \int \mathcal{D}_A \det\left\{\frac{\partial G[A]}{\partial \alpha}\right\} e^{-\frac{1}{2\xi}\int d^4x_E G^2[A]} e^{-S_E[A]}, \end{aligned} \tag{5.19}$$

where ξ is an arbitrary parameter that should **not** show up in the result. This path integral now looks similar to those we had for scalars/fermions, but with the added determinant which complicates things. To deal with this complication, we utilize the property of Grassmann variables and add an integral over the Grassmann fields c and $\overline{c}$ such that

$$Z_{\text{gf}} = \int \mathcal{D}_A \mathcal{D}_c \mathcal{D}_{\overline{c}} \exp\left(-\frac{1}{2\xi}\int d^4x_E G^2[A] - S_E[A] - \int d^4x_E \overline{c}\frac{\partial G[A]}{\partial \alpha} c\right). \tag{5.20}$$

We call these auxiliary Grassmann fields *Faddeev-Popov ghosts*, and are merely incorporated as a mathematical trick to compute the path integral (**not** physical fields). These auxiliary fields must fulfil periodic boundary conditions like scalar fields do (though unlike fermions), which renders the full Euclidean action of QED in practice as

$$S_E = S_{\text{matter}} + S_{\text{gauge}} + S_{\text{gf}} + S_{\text{ghosts}}, \tag{5.21}$$

where S_{matter} constitutes the matter terms (associated to the fermionic fields),

$$S_{\text{gauge}} = \frac{1}{2}\int d^4x_E F_{ab} F_{ab}, \tag{5.22a}$$

$$S_{\text{gf}} = \frac{1}{2\xi}\int d^4x_E G^2[A], \tag{5.22b}$$

$$S_{\text{ghosts}} = \int d^4x_E\, \overline{c}\frac{\partial G[A]}{\partial \alpha} c. \tag{5.22c}$$

§5.2.2 Solving the U(1) Path Integral

At this point, we now have the tools to solve the path integral over gauge fields in QED. Explicitly, this $U(1)$ path integral is written as

$$Z = \int \mathcal{D}_A \mathcal{D}_{\bar{c}} \mathcal{D}_c e^{-S_{\text{gauge}} - S_{\text{gf}} - S_{\text{ghosts}}}. \tag{5.23}$$

To start with, we need to choose a particular gauge condition to evaluate this, perferebly one which makes our lives the easiest. The canonical choice is the Landau gauge:

$$\begin{aligned} G[A] &= \partial_a A_a, && (5.24)\\ \Rightarrow \quad A_a \quad &\rightarrow \quad A_a - \partial_a \alpha, && (5.25) \end{aligned}$$

which then gives us that

$$\begin{aligned} S_{\text{ghosts}} &= \int d^4 x_E \, \bar{c} \frac{\partial G[A]}{\partial \alpha} c \\ &= \int d^4 x_E \, \bar{c} \partial_a \frac{\partial A}{\partial \alpha} c \\ &= \int d^4 x_E \, \partial_a \bar{c} \partial_a c. \end{aligned} \tag{5.26}$$

In this gauge, S_{ghosts} no longer depends on the gauge field so the partition function is separable into

$$Z = Z_A \times Z_{\text{ghost}}, \tag{5.27a}$$

$$\text{where} \quad Z_A = \int \mathcal{D}_A e^{-S_{\text{gauge}} - S_{\text{gf}}}, \quad Z_{\text{ghost}} = \int \mathcal{D}_{\bar{c}} \mathcal{D}_c e^{-S_{\text{ghosts}}}. \tag{5.27b}$$

The gauge transformation function $\alpha(x)$ must be periodic such that $\alpha(\tau = 0, \boldsymbol{x}) = \alpha(\tau = \beta, \boldsymbol{x})$, by construction of the gauge invariant Lagrangian. As such, the gauge fields are also periodic in the timelike direction which implies that they have associated Matsubara frequencies $K_0 = 2\pi n T$ in Fourier space. The gauge field term in the partition function is then

$$\begin{aligned} Z_A &= \int \mathcal{D}_{\tilde{A}} \exp\left(-\frac{1}{2\beta V} \sum_{\boldsymbol{K}} \tilde{A}_a(\boldsymbol{K}) \left[K^2 \delta_{ab} - K_a K_b + \frac{1}{\xi} K_a K_b \right] \tilde{A}_b(-\boldsymbol{K}) \right) \\ &= \prod_{\boldsymbol{K}} \det \left\{ K^2 \delta_{ab} - K_a K_b + \frac{1}{\xi} K_a K_b \right\}^{-\frac{1}{2}}. \end{aligned} \tag{5.28}$$

To further simplify this expression, we will need to compute the determinant, for which we define the matrix

$$M_{ab} \equiv K^2 \delta_{ab} - K_a K_b + \frac{1}{\xi} K_a K_b. \tag{5.29}$$

We decompose this matrix into 2 projectors

$$P_{ab}^{(T)} = \delta_{ab} - \frac{K_a K_b}{K^2}, \tag{5.30a}$$

$$P_{ab}^{(L)} = \frac{K_a K_b}{K^2}, \tag{5.30b}$$

which obey the identities

$$P_{ab}^{(T)} P_{bc}^{(L)} = 0, \tag{5.31a}$$
$$P_{ab}^{(T)} P_{bc}^{(T)} = P_{ac}^{(T)}, \tag{5.31b}$$
$$P_{ab}^{(L)} P_{bc}^{(L)} = P_{ac}^{(L)}. \tag{5.31c}$$

The trace values of these projectors can then be computed as

$$\mathrm{Tr}\Big\{P_{ab}^{(T)}\Big\} = 4 - 1 = 3, \tag{5.32a}$$
$$\mathrm{Tr}\Big\{P_{ab}^{(T)}\Big\} = 1, \tag{5.32b}$$

which tells us that the K^2 eigenvalue as multiplicity 3, whereas K^2/ξ has multiplicity 1. The result is that

$$\det\{M_{ab}\} = (K^2)^3 \left(\frac{K^2}{\xi}\right)^1 . \tag{5.33}$$

Plugging this into the partition function gives

$$Z_A = \exp\left(-\frac{1}{2}\sum_K \big[4\ln\big(K^2\big) + \ln(\xi)\big]\right). \tag{5.34}$$

Now for the ghost fields, we can also utilize the Fourier transform to get

$$\begin{aligned} Z_{\text{ghost}} &= \int \mathcal{D}_{\bar{c}}\mathcal{D}_c \exp\left(-\frac{1}{\beta V}\sum_K \bar{c}(K)K^2 c(K)\right) \\ &= \prod_K K^2 \\ &= \exp\left(\sum_K \ln K^2\right). \end{aligned} \tag{5.35}$$

The total partition function of the $U(1)$ gauge theory is then

$$Z = \exp\left(-\frac{1}{2}\sum_K \big[4\ln\big(K^2\big) + \ln(\xi)\big] + \sum_K \ln K^2\right), \tag{5.36}$$

for which in the large volume limit, we convert the sum over K to an integral and dimensionally regularize the ξ constant term away (because it poses no logarithmic divergence) which gives

$$\boxed{Z = \exp\left[-\sum_K \ln\big(K^2\big)\right]}, \tag{5.37}$$

which is exactly equal to the partition function of two free, real and massless scalar fields! As such, the $U(1)$ gauge field pressure is just

$$p(T) = 2p_{\text{free}}(m = 0, T) = \frac{2\pi^2 T^2}{90}, \tag{5.38}$$

which is in fact the pressure for perfect *blackbody radiation*. We also see that the $U(1)$ gauge field has two physical degrees of freedom.

§5.2.3 The Temporal-Axial Gauge

In this lecture, we will be looking at solving the path integral in the *temporal-axis gauge* (TAG).

> **Note**: Remember that performing the path integral in a different gauge does not change any of the physics since the theory is gauge **invariant**. However, it does give us different insights into the interpretations of the results and sometimes more intuitive equations to help with understanding.

We do this for the $U(1)$ gauge field, which we will see leads to a phenomenon known as the *Casimir effect*. First recall that our action (ignoring the matter associated fields) consists of the terms:

$$S_{\text{gauge}} = \frac{1}{2}\int d^4x_E F_{ab}F_{ab}, \tag{5.39a}$$

$$S_{\text{gf}} = \frac{1}{2\xi}\int d^4x_E G^2[A], \tag{5.39b}$$

$$S_{\text{ghosts}} = \int d^4x_E \, \bar{c}\frac{\partial G[A]}{\partial\alpha}c. \tag{5.39c}$$

The TAG condition is then written as

$$G^j[A] = -A_0, \tag{5.40}$$

with the sign is simply convention. In the TAG, since the gauge field transforms under gauge transformations as $A_a \to A_a - \partial_a\alpha$, we have that

$$S_{\text{ghost}} = \int d^4x_E \, \bar{c}\partial_0 c. \tag{5.41}$$

We once again have that S_{ghost} does not depend on the gauge field, so the partition function is separable into a gauge field dependent term and the ghost term:

$$Z = Z_A \times Z\text{ghost}, \tag{5.42a}$$

$$\text{where} \quad Z_A = \int \mathcal{D}_A e^{-S_{\text{gauge}}-S_{\text{gf}}}, \quad Z_{\text{ghost}} = \int \mathcal{D}_{\bar{c}}\mathcal{D}_c e^{-S_{\text{ghosts}}}. \tag{5.42b}$$

The path integral over temporal gauge fields A_0 can be performed for arbitrary gauge parameter ξ, for which we can send $\xi \to 0$ such that the gauge fixing partition function term becomes

$$\lim_{\xi\to 0} e^{-S_{\mathrm{gf}}} = \lim_{\xi\to 0} e^{-\frac{1}{2\xi}\int d^4x_E A_0^2} \to \prod_x \delta[A_0(x)]. \tag{5.43}$$

As for the gauge (field-strength tensor) term, we first consider a decomposition of the quadratic field-strength tensor term into spatial and temporal components

$$\begin{aligned} \frac{1}{2}F_{ab}F_{ab} &= F_{0j}F_{0j} + \frac{1}{2}F_{jk}F_{jk} \\ &= \partial_0 A_j \partial_0 A_j + \frac{1}{2}F_{jk}F_{jk} - 2\partial_j A_0 \partial_0 A_j + (\partial_j A_0)^2 . \end{aligned} \tag{5.44}$$

Integrating this with the gauge fixing term forces all A_0 terms to vanish (by the delta-function), which leaves

$$Z_A = \int \mathcal{D}_{A_j} \exp\left(-\frac{1}{2}\int d^4x_E \left[\partial_0 A_j \partial_0 A_j + \frac{1}{2}F_{ij}F_{ij}\right]\right), \tag{5.45}$$

where the measure $\mathcal{D}_{A_j}$ indicates that the path integral only considers the spatial components of the gauge field. Since we have quadratic terms again, we take the Fourier transform of the spatial gauge field terms

$$A_j(x) = \frac{1}{\beta V}\sum_{\boldsymbol{K}} e^{i\boldsymbol{K}\cdot x}\tilde{A}_j(\boldsymbol{K}), \tag{5.46}$$

where $\boldsymbol{K}$ is still the full 4-dimensional wave-vector $\boldsymbol{K} = (\omega_n, \boldsymbol{k})$, containing the Matsubara frequencies. Plugging this into the partition function gives

$$Z_A = \int \mathcal{D}\tilde{A}_i \exp\left[-\frac{1}{2}\sum_{\boldsymbol{K}} \tilde{A}_i(-K)\left[\omega_n^2\delta_{jk} + \mathbf{k}^2\delta_{ij} - k_i k_j\right]\tilde{A}_i(K)\right]. \tag{5.47}$$

Recall that in the Landau gauge, we arrived at a similar expression in Eq. (5.13) which was divergent since it had a zero eigenvalue with vector K_a. In the current case however, we see that the partition is now

$$Z_A = \prod_{\boldsymbol{K}} \det\left\{K^2\delta_{ij} - k_i k_j\right\}^{-\frac{1}{2}}, \tag{5.48}$$

for which we introduce two projectors onto orthogonal spaces

$$T_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}, \tag{5.49a}$$

$$L_{ij} = \frac{k_i k_j}{k^2}, \tag{5.49b}$$

we refer to as the *transverse* and *longitudinal projectors* in reference to their spatial orientations. Written in terms of these projectors, the matrix elements are then

$$K^2\delta_{ij} - k_i k_j = K^2 T_{ij} + \omega_n^2 L_{ij}, \tag{5.50}$$

making it clear that this matrix has **no** zero eigenvalues. So Z_A in the TAG is well-defined. Note that $\mathrm{Tr}\{T_{ij}\} = 2$ and $\mathrm{Tr}\{L_{ij}\} = 1$, so we have that the matrix has eigenvalue K^2 with multiplicity 2, and eigenvalue ω_n^2 with multiplicity 1. The result is:

$$Z_A = \exp\left[-\frac{1}{2}\sum_{\boldsymbol{K}} 2\ln\left(K^2\right) - \frac{1}{2}\sum_{\boldsymbol{K}} \ln\left(\omega_n^2\right)\right]. \tag{5.51}$$

The ghost partition function just has a single temporal derivative, so it results in

$$Z_{\text{ghost}} = e^{\sum_{\boldsymbol{K}} \ln(\omega_n)}, \tag{5.52}$$

which results in the total partition function

$$\boxed{Z = Z_A \times Z\text{ghost} = \exp\left[-\sum_{\boldsymbol{K}} \ln\left(K^2\right)\right]}, \tag{5.53}$$

which is exactly the result found from performing the computation in the Landau gauge. We note that the longitudinal contributions cancel in the final result, which is indicative of only transverse photons left as physically observed in the theory. This is of course known to be true in electrodynamics.

§5.2.4 The Casimir Effect

The *Casimir effect* arises in systems that have non-trivial spatial boundary conditions where in particular, we consider the case where 2 infinitely large parallel conducting plates experience a force sheerly due to vacuum fluctuations predicted by QED. To see this force arise, we consider the pressure p, and free energy Ω defined as

$$p = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}, \tag{5.54a}$$

$$\Omega = -\frac{\ln Z}{\beta V}. \tag{5.54b}$$

We set the non-trivial boundary conditions (conducting plates) at $z = 0$ and $z = L$, so that $\boldsymbol{E}_{\parallel}$ and $\boldsymbol{B}_{\perp}$ vanish there. This also implies that

$$A(t, x, y, z = 0) = A(t, x, y, z = L) = 0, \tag{5.55}$$

which then grants Fourier transform

$$\begin{aligned} A(z) &= \frac{1}{L}\sum_{k_z} e^{ik_z z}\tilde{A}(k_z) \\ &= \frac{1}{L}\sum_{k_z}\left[\cos(k_z z)\,\mathrm{Re}\left\{\tilde{A}(k_z)\right\} - \sin(k_z z)\,\mathrm{Im}\left\{\tilde{A}(k_z)\right\}\right] \\ &= -\frac{1}{L}\sum_{k_z}\sin\left(\frac{m\pi z}{L}\right)\mathrm{Im}\left\{\tilde{A}(k_z)\right\}, \end{aligned} \tag{5.56}$$

where $m \in \mathbb{Z}$. Recall that for free scalar field theory (without non-trivial boundary conditions), we would have path integrals of the form

$$\begin{aligned}\int \mathcal{D}_\phi e^{-\frac{1}{2}\int d^4x_E m^2\phi^2(x)} &\sim \int \mathcal{D}_\phi e^{-\frac{1}{2}\sum_{\boldsymbol{k}} m^2\tilde{\phi}^*\tilde{\phi}} \\ &= \int \mathcal{D}_{a_k}\mathcal{D}_{b_k} e^{-\frac{m^2}{2}\sum_{k=0}^\infty\left[a_k^2+b_k^2\right]} \\ &= \prod_{k=0}^\infty\left(\sqrt{\frac{\pi}{m^2}}\right)^2 \\ &= \prod_{k=-\infty}^\infty\sqrt{\frac{\pi}{m^2}} \\ &= \exp\left(-\frac{1}{2}\sum_k \ln\frac{\pi}{m^2}\right).\end{aligned} \tag{5.57}$$

However for the case of non-trivial boundary conditions, the real field components must vanish so we are left instead with just

$$\int \mathcal{D}_{b_k} e^{-\frac{m^2}{2}\sum_{k=0}^\infty b_k^2} = \exp\left(-\frac{1}{4}\sum_k \ln\frac{\pi}{m^2}\right), \tag{5.58}$$

so the parition function must be adjusted such that

$$Z = \exp\left(-\frac{1}{4}\sum_{\boldsymbol{K}} 2\ln\big(K^2\big)\right) \tag{5.59}$$

$$\begin{aligned}\Rightarrow\quad \ln Z &= -\frac{1}{2}\sum_{\omega_n}\sum_{k_\perp}\sum_{k_z}\ln\left(\omega_n^2+k_\perp^2+k_z^2\right) \\ &= -\sum_{k_\perp}\sum_{k_z}\left[\frac{\beta\sqrt{k_\perp^2+k_z^2}}{2}+\ln\left(1-e^{-\beta\sqrt{k_\perp^2+k_z^2}}\right)\right],\end{aligned} \tag{5.60}$$

where $k_\perp^2 = k_x^2 + k_y^2$. Considering just the low-temperature limit ($\beta/m \gg 1$), we have

$$\ln Z \approx -\frac{\beta}{2}\sum_{k_\perp}\sum_{k_z}\sqrt{k_\perp^2+k_z^2} \tag{5.61}$$

$$= -\frac{\beta}{2}\sum_{k_\perp}\sum_{k_z}\sqrt{k_\perp^2+\left(\frac{m\pi}{L}\right)^2}. \tag{5.62}$$

Since we have assumed that the plates are infinitely large, $k_\perp$ becomes continuous in this limit which renders the partition function as

$$\begin{aligned}\ln Z &= -\frac{\beta V_\perp}{2}\sum_{m=-\infty}^\infty\int\frac{d^2k_\perp}{(2\pi)^2}\sqrt{k_\perp^2+\left(\frac{m\pi}{L}\right)^2} \\ &= \frac{\beta V_\perp}{2}\sum_{m=-\infty}^\infty\frac{1}{6\pi}\left(\frac{m^2\pi^2}{L^2}\right)^{\frac{3}{2}} = \beta V_\perp\sum_{m=1}^\infty\frac{1}{6\pi}\left(\frac{m^2\pi^2}{L^2}\right)^{\frac{3}{2}},\end{aligned} \tag{5.63}$$

where we utilized the identity in Eq. (2.28). To further simplify this, we employ the use of the Riemann-zeta function to get

$$\begin{aligned} \ln Z &= \frac{\beta V_\perp \pi^2}{6L^3}\zeta(-3) \\ &= \frac{\beta V_\perp \pi^2}{720L^3}. \end{aligned} \tag{5.64}$$

> **Note**: The Riemann-zeta function appears here because $\zeta(x)$ when analytically continued to negative arguments, is essentially the same as performing dimensional regularization.

The Casimir pressure of this system is then

$$\boxed{p = \frac{1}{\beta V_\perp}\frac{\partial \ln Z}{\partial L} = -\frac{\pi^2}{240L^4}}. \tag{5.65}$$

We see that the pressure is negative, which leads to an attractive force between the boundary plates known as the Casimir effect. In SI, units this force per unit area is given as

$$p = \frac{F}{V_\perp} = -\frac{\pi^2 \hbar c}{240L^4} \approx -1.2 \times 10^{-27}\mathrm{N\ m^2\ L^{-4}}. \tag{5.66}$$

We see that this is an extremely small force, where even at nano scales ($L \sim 10^{-9}$), the force would still only have a magnitude of $\sim 10^{-9}$ N. This is strictly a prediction of QFT and has indeed been experimentally verified (Mohideen and Roy 1998, PRL 81 4549).

§5.2.5 The Anomalous Electron Magnetic Moment

One of the key tests that affirmed the success of QED was the precision measurement of the anomalous magnetic moment of the electron (i.e. the quantity $g-2$). In this section, we will be computing the g-factor, g from the QED formalism we have developed. Recall that our action for this theory was given by

$$S_E = S_{\text{matter}} + S_{\text{gauge}} + S_{\text{gf}} + S_{\text{ghosts}}, \tag{5.67}$$

where

$$S_{\text{matter}} = \int d^4x_E \overline{\psi}\left(\not{D} + m\right)\psi, \tag{5.68a}$$

$$S_{\text{gauge}} = \frac{1}{4e^2}\int d^4x_E F_{\mu\nu}F_{\mu\nu}, \tag{5.68b}$$

$$S_{\text{gf}} = \frac{1}{2\xi}\int d^4x_E G^2[A], \tag{5.68c}$$

$$S_{\text{ghosts}} = \int d^4x_E\, \bar{c}\frac{\partial G[A]}{\partial \alpha}c, \tag{5.68d}$$

with $\not{D} = D_\mu \gamma_E^\mu$ and $D_\mu = \partial_\mu + iA_\mu(x)$ is the covariant derivative.

Note: In this section, we use Greek letters even for Euclidean indices (i.e. raising and lowering the indices bears no consequence to the expressions) so that we free up Latin letters for color indices.

This action is invariant under $U(1)$ local gauge transformations, for which the gauge field transforms as $A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha(x)$. For convenience, we will scale the gauge ($A_\mu \to eA_\mu$) such that

$$S_{\text{matter}} + S_{\text{gauge}} = \int d^4x_E \left[\overline{\psi} \left(\not{\partial} + ie\not{A} \right) \psi + \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \right], \tag{5.69}$$

which is conventional in QED calculations. We computed the Dirac fermion Hamiltonian in Eq. (4.41), written as

$$H = \int d^3x\, \overline{\psi} \left(-i\gamma^j \partial_j + m \right) \psi, \tag{5.70}$$

where γ^j here denotes the **Minkowski** gamma matrices. To instead use the Euclidean gamma matrices, we write

$$H = \int d^3x\, \overline{\psi} \left(\gamma_E^j \partial_j + m \right) \psi. \tag{5.71}$$

This classical Hamiltonian is for a single Dirac fermion, however what we actually want in QED is to couple the Dirac field to the gauge field. This is done in S_{matter}, so we can see the classical Hamiltonian contribution to QED is given as

$$\Delta H = ie \int d^3x\, \overline{\psi} \not{A} \psi = ie \int d^3x\, \overline{\psi} \gamma_E^\mu \psi A_\mu. \tag{5.72}$$

To simplify this, we can consider a specific gauge fixing condition where $A_0 = 0$, which simplifies the classical contribution to

$$\Delta H = ie \int d^3x\, \overline{\psi} \gamma_E^j \psi A_j. \tag{5.73}$$

In the limit where the gauge field is small (matter dominated theory), the classical fermions will satisfy the Dirac equation

$$\left(i\gamma^\mu \partial_\mu - m \right) \psi = 0, \tag{5.74}$$

since any contribution that comes from the coupling from gauge fields and photons would be an additional contribution of order A_μ, which we take as small in amplitude. Considering just time-independent Dirac fields ($\psi = \psi(\boldsymbol{x})$) then leaves us with

$$m\psi = -\gamma_E^j \partial_j \psi, \tag{5.75a}$$

$$m\overline{\psi} = \left(\partial_j \overline{\psi} \right) \gamma_E^j. \tag{5.75b}$$

With some algebra, we find that

$$\boxed{\overline{\psi}\gamma_E^j\psi = \frac{1}{2m}\left[\left(\partial_k\overline{\psi}\right)\gamma_E^k\gamma_E^j - \overline{\psi}\gamma_E^j\gamma_E^k\left(\partial_k\psi\right)\right]}, \tag{5.76}$$

which is known as the *Gordon identity.* We can further simplify this identity by using the relations

$$\begin{aligned}\gamma_E^j\gamma_E^k &= \frac{1}{2}\left\{\gamma_E^j, \gamma_E^k\right\} + \frac{1}{2}\left[\gamma_E^j, \gamma_E^k\right] \\ &= \delta^{jk} - i\sigma^{jk},\end{aligned} \tag{5.77}$$

where $\sigma^{jk} \equiv \frac{i}{2}\left[\gamma_E^j, \gamma_E^k\right]$. Plugging this in to the Gordon identity then gives

$$\overline{\psi}\gamma_E^j\psi = \frac{1}{2m}\left[\left(\partial_j\overline{\psi}\right)\psi - \overline{\psi}\partial_j\psi - i\partial_k\left(\overline{\psi}\sigma^{kj}\psi\right)\right], \tag{5.78}$$

which can be put back into the classical Hamiltonian to give

$$\Delta H = \frac{ie}{2m}\int d^3x\left[\left(\partial_j\overline{\psi}\right)\psi - \overline{\psi}\partial_j\psi\right]A_j + \frac{e}{2m}\int d^3x\partial_k\left(\overline{\psi}\sigma^{kj}\psi\right)A_j. \tag{5.79}$$

The second term in the expression above is in fact the spin-orbit (SO) coupling term in the Hamiltonian, which can be rewritten as

$$\begin{aligned}\Delta H^{\mathrm{SO}} &= \frac{e}{2m}\int d^3x\partial_k\left(\overline{\psi}\sigma^{kj}\psi\right)A_j \\ &= -\frac{e}{4m}\int d^3x\,\overline{\psi}\sigma^{jk}\psi F_{jk}.\end{aligned} \tag{5.80}$$

Considering an constant, uni-axial external magnetic field such that the only non-trivial term in the field-strength tensor is $F_{12} = B_3$, we then have

$$\Delta H^{\mathrm{SO}} = -\mu_B\int d^3x\,\overline{\psi}\sigma^{12}\psi B_3, \tag{5.81}$$

where $\mu_B = e/(2m)$. The the Dirac matrix representation for the gamma matrices, we have that

$$\sigma^{12} = \frac{i}{2}\left[\gamma_E^1, \gamma_E^2\right] = -\begin{bmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{bmatrix}, \tag{5.82}$$

with σ_3 being the z Pauli matrix. The spin operator for fermions (spin-$\frac{1}{2}$ particles) is given as

$$S_3 = \frac{1}{2}\begin{bmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{bmatrix}, \tag{5.83}$$

allowing us to then write the classical spin-orbit Hamiltonian as

$$\boxed{\Delta H^{\mathrm{SO}} = 2\mu_B \int d^3x\, \overline{\psi} \left(\boldsymbol{S} \cdot \boldsymbol{B}\right) \psi}. \tag{5.84}$$

The factor of 2 out front can be principle deviate from the value 2, for which it is then in general referred to as the *g-factor* with symbol g. The treatment of the theory we have performed up till now (albeit containing Dirac fields and being relativistic), is not yet a QFT since no path integrals were taken. Hence, we can still consider this as a classical approximation (resulting in $g = 2$). Quantizing this theory will result in $g \neq 2$, for which this is then (for historic reasons) referred to as the *anomalous contribution* to the magnetic moment. We will see this now.

Recall that the Hamiltonian responsible for the magnetic moment of the electron is given in Eq. (5.72), for which $\not{A}$ also appears in the matter term of the action

$$S_{\text{matter}} = \int d^4x_E \overline{\psi} \left(\not{\partial} + ie\not{A}\right) \psi. \tag{5.85}$$

Including the QFT loop corrections will then change the effective fermion-gauge field coupling, which we will see arise using perturbation theory where we treat $e \ll 1$. Recall that the slash on the gauge field denotes $\not{A} = \gamma_\mu^E A_\mu$, which gives us that explicitly, the fermion-photon vertex is simply the integrand

$$ie\overline{\psi}\gamma_\mu^E A_\mu \psi. \tag{5.86}$$

To extend this to QFT, we would get a *vertex operator* (responsible for the electron and photon interaction) which arises from the taking the expectation value

$$\Gamma_\mu(x_1, x_2, x_3) = \int \mathcal{D}_{\overline{\psi}} \mathcal{D}_\psi \mathcal{D}_A e^{-S_E} \left[\overline{\psi}(x_1) A_\mu(x_2) \psi(x_3)\right] = \langle \overline{\psi}(x_1) A_\mu(x_2) \psi(x_3) \rangle_{\text{full}}. \tag{5.87}$$

We can expand this into a perturbation series as in Eq. (3.8), which gives us the lowest non-trivial order term as

$$\begin{aligned} \Gamma_\mu^{(1)}(x_1, x_2, x_3) &= -\langle S_{\text{matter}} \overline{\psi}(x_1) A_\mu(x_2) \psi(x_3) \rangle \\ &= -ie\gamma_\nu^E \int d^4x_E \langle \overline{\psi}(x) A_\nu(x) \psi(x) \overline{\psi}(x_1) A_\mu(x_2) \psi(x_3) \rangle. \end{aligned} \tag{5.88}$$

To evaluate this, we utilize the Wick's theorem which reduces the expression under the integral to two free fermion propagators $\langle \psi\overline{\psi} \rangle$ and a free photon propagator $\langle A_\mu A_\nu \rangle$. The integral over x_E then enforces momentum conservation. Since these terms arise repeatedly in the perturbation series, we can once again lighten the notation by introducing the connected, amputated vertex function, which to leading order is simply written as

$$\Gamma_\mu^{(1),\text{conn,. amp.}} = ie\gamma_\mu^E. \tag{5.89}$$

This results in the leading order Hamiltonian as

$$\Delta H^{(1)} = \int d^3x\, \overline{\psi} A_\mu \Gamma_\mu^{(1),\text{conn,. amp.}} \psi. \tag{5.90}$$

To get the full Hamiltonian correction, we simply replace $\Gamma_\mu^{(1),\text{conn},\text{. amp.}}$ with $\Gamma_\mu^{\text{conn},\text{. amp.}}$. From here, in order to compute the QED corrections to the g-factor, we will be required to calculate the corrections to the fermion-photon vertex. It will once again be advantageous to work in Fourier space, where we consider the Fourier transformed vertex function $\tilde{\Gamma}_\mu(p_1, p')$. Notice that this vertex function is only a function of 2 external momenta, precisely because momentum is conserved (we choose these to describe the two electron momenta). We find that in general, it is possible to decompose the full vertex function (in Fourier space) into

$$\tilde{\Gamma}_\mu(p, p') = ie\gamma_\mu F_1(q^2) + \frac{ie}{2m}\sigma_{\mu\nu}q_\nu F_2(q^2), \tag{5.91}$$

where p, p' are the chosen 4-momenta of incoming and outgoing electrons, and $q = p' - p$ is the photon 4-momentum. The functions $F_1(q^2)$ and $F_2(q^2)$ are known as *form factors*, which in the first-order vertex function are

$$F_1(q^2) = 1, \tag{5.92a}$$

$$F_2(q^2) = 0, \tag{5.92b}$$

We can use the Gordon identity in Eq. (5.76) to replace the gamma matrix in the connected vertex, which in Fourier space gives

$$\bar{\psi}(p')\,\gamma_\mu^E \psi(q) = -\frac{1}{2m}\bar{\psi}(p')\left[i\left(p'_\mu + p_\mu\right) - \sigma_{\mu\nu}p_\nu\right]\psi(q). \tag{5.93}$$

Plugging this back into the vertex function results in

$$\boxed{\tilde{\Gamma}_\mu(p, p') = \frac{e}{2m}(p_\mu + p'_\mu)F_1(p^2) + \frac{ie}{2m}\sigma_{\mu\nu}q_\nu\left[F_1(q^2) + F_2(q^2)\right]}. \tag{5.94}$$

We notice that only the second term above would contribute corrections pertaining to the magnetic field, which still implies that there is a relation between g and the form factors. The energy change from spin-orbit coupling is computed from the expectation value $\langle \Delta H \rangle$, for which since we assume the external magnetic field is constant, we can take that the photon momentum is zero ($q = 0$) which leaves us with

$$g = 2\left[F_1(0) + F_2(0)\right]. \tag{5.95}$$

As common in QFTs, it turns that that $F_1(0)$ and $F_2(0)$ are divergent but renormalizable by the introduction of counter-terms in the Lagrangian (particularly in the electron charge, $e \to e_{\text{phys}} + \delta e$). If we choose the on-shell (OS) renormalization scheme, it works out that $F_1(0) = 1$, which gives

$$g - 2 = 2F_2^{\text{OS}}(0), \tag{5.96}$$

leaving us the task of computing $F_2^{\text{OS}}(0)$ from the fermion-photon vertex. If we approach this perturbatively, we saw that $F_2(0) = 0$ in the first-order vertex function and in fact is also trivial in the second-order vertex function (because it consists of an odd number of A_μ terms). So the next lowest non-trivial order would be order 3. This is written explicitly as

$$\begin{aligned}\Gamma_\mu^{(3)}(x_1, x_2, x_3) = \frac{(-ie)^3}{3!}\int d^4x d^4y d^4z \langle \overline{\psi}(x)\not{A}(x)\psi(x)\overline{\psi}(y)\not{A}(y)\psi(y)\overline{\psi}(z)\not{A}(z)\psi(z) \\ \times \overline{\psi}(x_1)A_\mu(x_2)\psi(x_3)\rangle.\end{aligned} \tag{5.97}$$

Since the expectation here is taken w.r.t the free theory, it is Gaussian and can be decomposed via Wick's theorem into 2-point function terms we define as

$$S_{x,y}^{\alpha,\beta} = \langle \psi_x^\alpha \overline{\psi}_y^\beta \rangle, \tag{5.98a}$$

$$G_{x,y}^{\mu,\rho} = \langle A_{x,\mu} A_{y,\rho} \rangle, \tag{5.98b}$$

where α and β are the spinor indices we now make explicit, and we have moved the spacetime coordinate arguments to indices (i.e. $\psi(x) = \psi_x$). Contracting all terms with the appropriate permutation factors then leaves us with

$$\Gamma_\mu^{(3)}(x_1, x_2, x_3) = ie^3 \int_{x,y,z} G_{y,z}^{\sigma\delta} G_{2,x}^{\mu\rho} S_{3,y} \gamma_\sigma S_{y,x} \gamma_\rho S_{x,z} \gamma_\delta S_{z,1}. \tag{5.99}$$

We can now Fourier transform this which gives

$$\tilde{\Gamma}_\mu^{(3)}(P, P') = ie^3 \int_K G_K^{\rho\sigma} G_Q^{\mu\rho} S_{P'} \gamma_\sigma S_{K+P'} \gamma_\rho S_{K+P} \gamma_\delta S_P \delta(Q - (P' - P)), \tag{5.100}$$

and lighten the notation once again with the amputated vertex function

$$\tilde{\Gamma}_\mu^{(3),\text{conn., amp.}}(P, P') = ie^3 \int_K G_K^{\rho\sigma} \gamma_\sigma S_{K+P'} \gamma_\mu S_{K+P} \gamma_\delta, \tag{5.101}$$

where we have omitted the momentum Q photon propagator $G_Q^{\mu\rho}$, and the external fermion propagator S_P and $S_{P'}$.

§5.3 Non-Abelian Gauge Theories

In the previous gauge theory we studied, the symmetry was a local $U(1)$ transformation $\phi(x) \to e^{i\alpha(x)}\phi(x)$. We have also seen that the $U(1)$ transformations are isomorphic to $SO(2)$ if we decompose the complex scalar field as $\phi(x) = (\phi_1 + i\phi_2)/\sqrt{2}$. In this section, we are going to generalize this to transformations of an $SO(N)$ gauge group. To do so, we can start by considering the $SO(3)$ gauge group with the 3-component $O(N)$-vector model which has the action

$$S_E = \int d^4x_E \left[\frac{1}{2} \partial_a \boldsymbol{\phi} \cdot \partial_a \boldsymbol{\phi} + \frac{m^2}{2} \boldsymbol{\phi} \cdot \boldsymbol{\phi} \right]. \tag{5.102}$$

This action is invariant under the global $SO(3)$ transformation

$$\phi_j \quad \to \quad R_{jk}(\alpha_1, \alpha_2)\phi_k(x), \tag{5.103}$$

where $R_{jk}(\alpha_1, \alpha_2)$ is the 3-dimensional rotation matrix with 2 rotation parameters (angles) α_1 and α_2. We could continue the analysis in $SO(3)$, however it would be more mathematically convenient to continue discussions in $SU(2)$ instead (whose Lie algebra is isomorphic). To do so, we consider the object

$$\Phi(x) = \frac{1}{2} \sum_{j=1}^{3} \phi_j(x) \sigma_j, \tag{5.104}$$

where σ_j are the Pauli matrices which are Hermitian, unitary and have $\mathrm{Tr}\{\sigma_j\sigma_k\} = 2\delta_{jk}$. As such, we have the identity

$$\begin{aligned}\mathrm{Tr}\big\{\partial_a\Phi(x)\partial_a\Phi^\dagger(x)\big\} &= \frac{1}{4}\partial_a\phi_j(x)\partial_a\phi_k(x)\,\mathrm{Tr}\{\sigma_j\sigma_k\}\\ &= \frac{1}{2}\partial_a\phi_j(x)\partial_a\phi_j(x).\end{aligned} \tag{5.105}$$

As such, we can rewrite the action as

$$S_E = \mathrm{Tr}\int d^4x_E\left[\partial_a\Phi(x)\partial_a\Phi^\dagger(x) + m^2\Phi(x)\Phi^\dagger(x)\right]. \tag{5.106}$$

The $SU(2)$ transformation on this new field object is then

$$\Phi(x) \quad\to\quad e^{i\alpha_j\sigma_j}\Phi(x), \tag{5.107a}$$
$$\Phi^\dagger(x) \quad\to\quad \Phi^\dagger(x)e^{-i\alpha_j\sigma_j}, \tag{5.107b}$$

because

$$\mathrm{Tr}\big\{e^{i\alpha_j\sigma_j}\Phi(x)\Phi^\dagger(x)e^{-i\alpha_j\sigma_j}\big\} = \mathrm{Tr}\big\{e^{-i\alpha_j\sigma_j}e^{i\alpha_j\sigma_j}\Phi(x)\Phi^\dagger(x)\big\} = \mathrm{Tr}\big\{\Phi(x)\Phi^\dagger(x)\big\}. \tag{5.108}$$

So the action is indeed invariant under global $SU(2)$ transformations. Now we are going to repeat the same procedure for local $SU(2)$ gauge transformations such that

$$\Phi(x) \quad\to\quad e^{i\alpha_j(x)\sigma_j}\Phi(x), \tag{5.109a}$$
$$\Phi^\dagger(x) \quad\to\quad \Phi^\dagger(x)e^{-i\alpha_j(x)\sigma_j}. \tag{5.109b}$$

It works out that the corresponding gauge-covariant derivative in the $SU(2)$ case (analogous to the $U(1)$ case) is given by

$$D_a = \partial_a + iA_a(x), \tag{5.110}$$

such that the gauge field here transforms as

$$A_a(x) \quad\to\quad U(x)A_a(x)U^\dagger(x) + i[\partial_a U(x)]U^\dagger(x), \tag{5.111a}$$
$$D_a\Phi(x) \quad\to\quad U(x)D_a\Phi(x), \tag{5.111b}$$

where $U(x) \equiv e^{i\alpha_j(x)\sigma_j}$, under local $SU(2)$ gauge transformations. We can also ask what the analog of the field-strength tensor (F_{ab}) is in $U(1)$ for $SU(2)$. To do so, we first write

$$F_{ab} = -i[D_a, D_b] = \partial_a A_b - \partial_b A_a, \tag{5.112}$$

which allows us to generalize this into $SU(2)$ such that now

$$F_{ab} = -i[D_a, D_b] = \partial_a A_b - \partial_b A_a + i[A_a, A_b]. \tag{5.113}$$

This indeed transforms appropriately as

$$F_{ab} \quad\to\quad UF_{ab}U^\dagger, \tag{5.114}$$

and has $\mathrm{Tr}\{F_{ab}F_{ab}\}$ invariant under local $SU(2)$ gauge transformations, for which the additional term $[A_a, A_b]$ in F_{ab} is now non-vanishing since the fields no longer commute. As such, this gauge fields are called *non-abelian* and the theory is then a *non-abelian gauge theory*. We conventionally include this non-abelian field-strength tensor into the action by

$$S_E = \mathrm{Tr} \int d^4x_E \left[\partial_a \Phi(x) \partial_a \Phi^\dagger(x) + m^2 \Phi(x) \Phi^\dagger(x) + \frac{1}{2g^2} F_{ab} F_{ab} \right], \tag{5.115}$$

where the factor $1/(2g^2)$ is convention and g is known as the coupling constant of the non-abelian field theory. We can also decompose the field-strength tensor into its vector components as

$$F_{ab}(x) = \frac{1}{2} \sum_{j=1}^{3} \sigma_j F^j_{ab}(x), \tag{5.116}$$

so we have that

$$\mathrm{Tr}\left\{ \frac{1}{2g^2} F_{ab} F_{ab} \right\} = \frac{1}{4g^2} F^j_{ab} F^j_{ab}, \tag{5.117}$$

with the indices running over $a, b = 1, 2, 3, 4$ and $j = 1, 2, 3$. Recall that a, b are the Euclidean Lorentz indices whereas j is referred to as the *color index*. Since we have used fairly general notation for the $N = 3$ case using the color index, we can easily extend this to the case for arbitrary N such that under $SU(N)$ transformations, the non-abelian gauge field transforms as already written above:

$$\boxed{A_a(x) \quad \to \quad U(x) A_a(x) U^\dagger(x) + i \left[\partial_a U(x)\right] U^\dagger(x)}, \tag{5.118}$$

$$\text{where} \quad U(x) = \exp\left(i \alpha_j(x) \lambda_j\right), \tag{5.119}$$

with λ_j being an element of $SU(N)$ and $j \in [1, N^2 - 1]$. The general non-abelian field strength tensor is also then decomposeable into generators of the $SU(N)$ group t_j, as

$$\boxed{F_{ab} = \sum_{j=1}^{N^2-1} F^j_{ab} t_j}, \tag{5.120}$$

in which the generators are normalized such that $\mathrm{Tr}\{t_j t_k\} = \frac{1}{2}\delta_{jk}$. These generators must of course the Lie algebra associated to $SU(N)$ which is

$$[t_j, t_k] = i f_{jkl} t_l, \tag{5.121}$$

where f_{jkl} is called the *structure tensor*, allowing these generators to have an $N \times N$ matrix representation. For $N = 2$, the structure tensor is simply the Levi-Civita tensor ϵ_{jkl}.

Note: The object F_{ab} and the set of F^j_{ab} carry the same information. F_{ab} is called the *fundamental representation* (usually written as an $N \times N$ matrix), while the set of $N^2 - 1$ numbers F^j_{ab}, is known the *adjoint representation* which can be thought of as components to the basis vectors t_j. These representations are related via

$$F^j_{ab} = 2\,\mathrm{Tr}\big\{F_{ab} t^j\big\}. \tag{5.122}$$

The general $SU(N)$ gauge theory is referred to as the *Yang-Mills theory.* The case where $N = 3$ is in fact the theory strong interactions known as quantum chromodynamics (QCD), with the name “chromo” due to the indices being called color, whereas the case where $N = 2$ is a theory for electroweak interactions known as the *Salam-Glashow-Weinberg model.*

To now solve the Yang-Mills theory, we once again focus on the non-abelian gauge field term which renders the action just as

$$S_E = \frac{1}{2g^2}\,\mathrm{Tr}\int d^4x_E F_{ab}F_{ab} = \frac{1}{4g^2}\int d^4x_E F^j_{ab}F^j_{ab}, \tag{5.123a}$$

$$\text{where}\quad F^j_{ab} = \partial_a A^j_b - \partial_b A^j_a - f_{jkl}A^k_a A^l_b, \tag{5.123b}$$

where a, b denoting the Euclidean Lorentz indices and j, k, l denoting the color indices. The partition function is then

$$Z = \int \mathcal{D}_A \exp\left(-\frac{1}{4g^2}\int d^4x_E F^j_{ab}F^j_{ab}\right). \tag{5.124}$$

To first consider a simpler problem, we can rescale the gauge field $A_a(x) \to gA_a(x)$ such that

$$Z = \int \mathcal{D}_A \exp\left(-\frac{1}{4}\int d^4x_E F^j_{ab}F^j_{ab}\right), \tag{5.125a}$$

$$\text{where}\quad F^j_{ab} = \partial_a A^j_b - \partial_b A^j_a - g f_{jkl}A^k_a A^l_b. \tag{5.125b}$$

In this new rescaled partition function, we can consider the case where $g = 0$ (referred to as *free Yang-Mills theory*), which leaves the field-strength tensor uncoupled in the gauge fields

$$F^j_{ab}[g = 0] = \partial_a A^j_b - \partial_b A^j_a, \tag{5.126}$$

and thus admits a factorization of the partition function into $N^2 - 1$ product terms

$$Z[g = 0] = \prod_{j=1}^{N^2-1} \int \mathcal{D}_A \exp\left(-\frac{1}{4}\int d^4x_E F^2\right) = \left(Z_{U(1)}\right)^{N^2-1}. \tag{5.127}$$

So we see that the $g = 0$ non-abelian partition function is simply a product of partition functions for an abelian gauge theory. This results in free Yang-Mills theories (and hence weak-coupling perturbation theories) to suffer from the gauge-orbit problems found in $U(1)$ theories. However, we will see that the coupled Yang-Mills theory is completely well defined for $g \neq 0$. To solve coupled Yang-Mills theories, we can once again employ the trick of Faddeev-Popov ghosts with a suitable gauge $G^j[A] = f^j$, chosen to simplify the calculations (a similar procedure to that done for QED). This results in a very familiar path integral

$$\begin{aligned} Z[g] &= \int \mathcal{D}_A\, \delta(G^j[A] - f^j)\, \det\left\{\frac{\partial G^j[A]}{\partial \alpha^j}\right\} e^{-S_E[A]} \\ &= \int \mathcal{D}_A \mathcal{D}_{\bar{c}} \mathcal{D}_c\, \delta(G^j[A] - f^j)\, \exp\left(-S_E[A] - \int d^4x_E\, \bar{c}^j \frac{\partial G^j[A]}{\partial \alpha^j} c^j\right). \end{aligned} \tag{5.128}$$

Integrating over the gauge fixings with Gaussian weight leads to

$$Z[g] = \int \mathcal{D}_A \mathcal{D}_{\overline{c}} \mathcal{D}_c e^{-S_E - S_{\text{ghost}} - S_{\text{gf}}}, \tag{5.129}$$

where

$$S_E = \frac{1}{4g^2} \int d^4x_E F^j_{ab} F^j_{ab}, \tag{5.130a}$$

$$S_{\text{ghost}} = \int d^4x_E \, \overline{c}^j \frac{\partial G^j[A]}{\partial \alpha^j} c^j, \tag{5.130b}$$

$$S_{\text{gf}} = \frac{1}{2\xi} \int d^4x_E G^j[A] G^j[A]. \tag{5.130c}$$

Once again, ξ is just an arbitrary parameter and must drop out of the physical observables derived in this theory.

Let us try solving this integral in the **Landau gauge**, where the gauge condition is given by $G^j[A] = \partial_a A^j_a$. In the fundamental representation, we still have that the gauge field transforms as

$$A_a(x) \quad \rightarrow \quad U(x) A_a(x) U^\dagger(x) + i \left[\partial_a U(x)\right] U^\dagger(x), \tag{5.131}$$

under the local gauge transformation $U(x) = e^{i\alpha^j(x) t^j}$. For small values of α, the transformation of the gauge field in the adjoint representation is given as

$$A^j_a(x) \quad \rightarrow \quad A^j_a(x) - \partial_a \alpha^j(x) - f^{jkl} \alpha^k(x) A^l_a(x). \tag{5.132}$$

Compared to the case of $U(1)$ transformations, the gauge transformation for $A_a(x)$ has an additional term $-f^{jkl}\alpha^k(x)A^l_a(x)$, which is itself dependent on the gauge field. As a consequence, the ghost term in the partition function will also include a gauge field term (unlike in the case for $U(1)$ transformations):

$$\begin{aligned} S_{\text{ghosts}} &= \int d^4x_E \, \overline{c} \frac{\partial G[A]}{\partial \alpha} c \\ &= \int d^4x_E \left[\partial_a \overline{c}^j \partial_a c^j + \partial_a \overline{c}^j f^{jkl} A^l_a c^k\right]. \end{aligned} \tag{5.133}$$

Once again rescaling the gauge field $A_a(x) \rightarrow g A_a(x)$ to allow considerations of the zero-coupling ($g = 0$) case, gives us the partition function

$$Z = \int \mathcal{D}_A \mathcal{D}_{\overline{c}} \mathcal{D}_c \exp\left[-\int d^4x_E \left(\underbrace{\frac{1}{4} F^j_{ab} F^j_{ab}}_{\text{Yang-Mills}} + \underbrace{\frac{1}{2\xi} \partial_a A^j_a \partial_a A^j_a}_{\text{gauge-fixing}} + \underbrace{\partial_a \overline{c}^j \partial_a c^j + \partial_a \overline{c}^j f^{jkl} A^l_a c^k}_{\text{ghost}}\right)\right]. \tag{5.134}$$

So unlike QED, the partition function now would have self-interaction due to the nonlinear $f^{jkl} A^k_a A^l_b$ term in the field-strength tensor. This makes computations complicated (even the one-loop calculations), involving gauge field vertices as well as vertices that couple the ghost field and the gauge field.

Appendices

Appendix A

Grassmann Variables

In normal arithmetic, the product of two numbers a and b following the commuting relation $ab = ba$. However, we can also consider anti-commuting numbers which follow the relation

$$ab = -ba. \tag{A.1}$$

Such anti-commuting numbers are known as *Grassmann numbers*. A property which follows from anti-commutation is *nilpotence*, which means

$$a^2 = a^3 = a^4 = \ldots = 0. \tag{A.2}$$

> **Note**: Grassmann numbers can be represented in terms of $2^n \times 2^n$ matrices.

Nilpotence implies that any Taylor expansion of a function of a Grassman number θ terminates after at the second order term:

$$\boxed{f(\theta) = c_0 + c_1\theta}. \tag{A.3}$$

where c_0 and c_1 are usual commuting coefficients (sometimes called c-numbers). Also, Grassmann variables satisfy the *Grassmann integration rule* for translationally invariant integrals. That is to say, if the integral is invariant under the transformation $\theta \to \theta + \eta$, then we have that

$$\boxed{\int d\theta f(\theta) = \int d\theta (c_0 + c_1\theta) = c_1}, \tag{A.4}$$

which implies that

$$\int d\theta = 0, \quad \int \theta d\theta = 1. \tag{A.5}$$

Note: The integration properties in Eq. (A.5) are identical to the derivative rules for variables of c-numbers, i.e.

$$\frac{\partial}{\partial x}1 = 0, \quad \frac{\partial}{\partial x}x = 1. \tag{A.6}$$

The proof of the Grassmann integral identity is as follows.

Proof. Considering the integral

$$F(c_0, c_1) = \int d\theta f(\theta), \tag{A.7}$$

the Taylor expansion property of a function of Grassmann variables gives us that

$$F(c_0, c_1) = \int d\theta \left(c_0 + c_1\theta\right). \tag{A.8}$$

Furthemore, since the integrand is a linear function of the c-variables c_0 and c_1, we can write the integral as

$$F(c_0, c_1) = \alpha c_0 + \beta c_1, \tag{A.9}$$

where α and β are also c-numbers. From this, we consider the consequence of translational invariance of this integral. Translating the Grassmann variable via the transformation $\theta \to \theta + \eta$, we get

$$\begin{aligned} \int d\theta \left(c_0 + c_1\theta\right) \quad \to \quad \int d\theta \left[c_0 + c_1(\theta + \eta)\right] &= F(c_0 + \eta c_1, c_1) \\ &= \alpha(c_0 + \eta c_1) + \beta c_1. \end{aligned} \tag{A.10}$$

Equating the above expression with the pre-translated integral gives

$$\begin{aligned} & \alpha c_0 + \beta c_1 = \alpha(c_0 + \eta c_1) + \beta c_1, \\ \Rightarrow \quad & \alpha\eta c_1 = 0, \\ \Rightarrow \quad & \alpha = 0, \end{aligned} \tag{A.11}$$

since we noted that $\eta \neq 0$ and $c_1 \neq 0$. Finally, we can arbitrarily normalize the remaining constant β to one, which result in

$$F(c_0, c_1) = \int d\theta f(\theta) = c_1. \tag{A.12}$$

□

An example of the application of these Grassmann variable properties, is when the integral function is a Gaussian of multiple Grassmann variables. We can see that

$$\int d\theta_0 d\theta_1 e^{-\theta_0 b\theta_1} = \int d\theta_0 d\theta_1 (1 - \theta_0 b\theta_1), \tag{A.13}$$

where b is a c-number. Using the integral properties, we get that

$$\int d\theta_0 d\theta_1 e^{-\theta_0 b \theta_1} = b \int \theta_0 d\theta_0 \int \theta_1 d\theta_1 = b. \tag{A.14}$$

This can be generalize to a Gaussian of $2N$ variables which gives

$$\prod_{j=1}^{N} \int d\theta_j d\phi_j e^{-\theta_j B_{ij} \phi_j} = \det(B_{ij}). \tag{A.15}$$